2000.03.06 ~

・[illegible]　・[illegible]

・サボる

・具体例

No. 1/9

Kikuchi On g_X

[illegible]サボる　指　D　SW

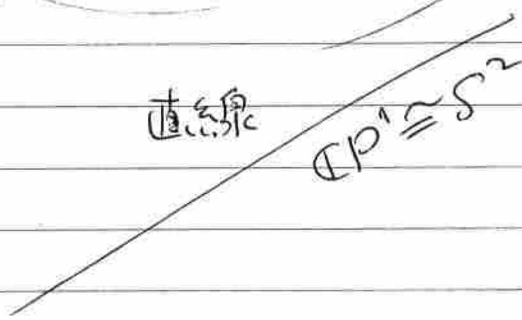
・ユーザーに徹する

・マニュアルも見ない

§1. g_X の定義

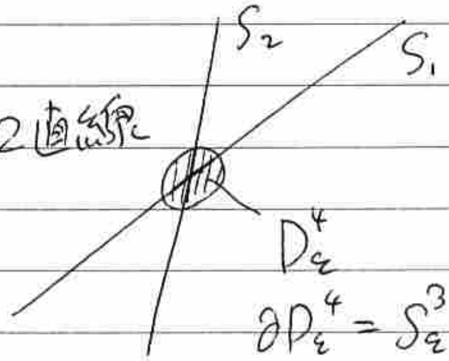
$$\begin{cases} C^\infty\text{-category} \\ \mathbb{Z}\text{-homology} \\ \text{(co-)} \end{cases}$$

$\mathbb{C}P^2$



直線　$\mathbb{C}P^1 \cong S^2$

$$H_2(\mathbb{C}P^2) = \mathbb{Z}[\mathbb{C}P^1]$$



S_2　S_1

2直線

D^4_ε

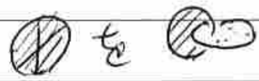
$\partial D^4_\varepsilon = S^3_\varepsilon$

$(S_1 \cup S_2) \cap \partial D^4_\varepsilon$

$=$

$S_2 \cap \partial D^4_\varepsilon$　$S_1 \cap \partial D^4_\varepsilon$

annulus



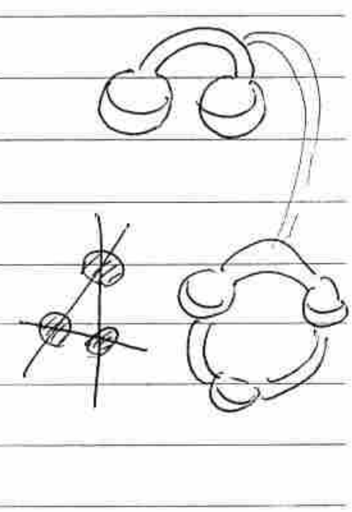
を　でおきかえても

homology class は変わらない。

$$2[\mathbb{C}P^1] = [S_1] + [S_2] = [\underset{\overset{\Vert}{S^2}}{S_1 \# S_2}]$$

同様に

$$3[\mathbb{C}P^1] = [S_1 \# S_2 \# S_3] = [T^2]$$



X : closed oriented 4-mfd,
$\pi_1(X) = 1$ for simplicity

$\xi \in \cancel{H_2(X;\mathbb{Z})}\ H_2(X)$

$$\xi = [S^2 \overset{\exists}{\to} X] \quad \left(\begin{array}{l} H_2(X) \cong \pi_2(X) \\ C^\infty \text{ 近似} \end{array}\right)$$

$$\cancel{\text{自己交叉解消}} \qquad = [\Sigma_g \overset{\exists}{\hookrightarrow} X] \quad \left(\text{自己交叉解消}\right)$$

genus g の閉曲面

$$= [\Sigma_{g+1} \overset{\exists}{\hookrightarrow} X] \quad \left(\begin{array}{l} X \# S^4 \cong X \\ \Sigma_g \# T^2 \cong \Sigma_{g+1} \end{array}\right)$$

$$g_X(\xi) := \min\{ g \mid \xi = [\Sigma_g \hookrightarrow X] \}$$

$$g_X : H_2(X) \to \mathbb{N} \cup \{0\} \quad C^\infty \text{ invariant of } X.$$

$\boxed{Q}$ $g_X(\xi) = ?$ for given X, ξ

$\boxed{e}$ $g_{\mathbb{CP}^2 \# \mathbb{CP}^2}(3,2) \overset{?}{=} 1$ $\left(\begin{array}{c}\cancel{\text{Fact}} \\ \cancel{\leq 1}\end{array}\right)$

$\cdot\ g_{\mathbb{CP}^2 \# \mathbb{CP}^2}(3,2) \leq 1$

$X = \mathbb{CP}^2 \# \mathbb{CP}^2$
$\xi = 3[\mathbb{CP}^1_1] \cancel{+} 2[\mathbb{CP}^1_2]$
$= [T^2] + [S^2]$
$= [T^2 \# S^2]$
$= [T^2]$

<u>Thom conjecture</u> (generalized, 本来は $X = \mathbb{CP}^2$)

X complex algebraic surface
$\cup$
C smooth algebraic curve

$\Rightarrow$

$$g_X([C]) = \text{genus}(C)$$

$$= \frac{\overset{\text{P.D.}}{K_X} C + C \cdot C}{2} + 1$$

canonical class of X
$\updownarrow$
$c_1(X)$

§2 Rohlin と q_X (1950s ~ ~~1960s~~)

Rohlin X spin $(w_2(X)=0)$ $\Longrightarrow$ $\sigma(X) \equiv_{(16)} 0$

X の signature

$$
\left\{
\begin{aligned}
&q_X : H_2(X) \times H_2(X) \to \mathbb{Z} \quad \text{intersection form (unimodular)} \Leftarrow \text{Poincaré duality}\\
&b_2(X) = q_X \text{ の rank}\\
&b_2^{\pm}(X) = q_X \text{ の } \pm \text{ の固有値の数}\\
&\sigma(X) = b_2^+(X) - b_2^-(X)\\
&\chi(X) = b_2^+(X) + b_2^-(X) + 2
\end{aligned}
\right.
$$

Rohlin の ⓘ

① ~~by Rohlin~~ homotopy theory ~~theoretic~~ $\left(\frac{3}{2}\sigma(X) \in \mathbb{Z} \mapsto 0 \in \mathbb{Z}_{24}: \ \pi_3(SOm) \to \pi_{m+3}(S^m) \to 0 \right)$ に帰着

② geometric topology ~~Arf inv + cobordism~~ $\left((\mathbb{C}P^2, z_2^s + z_1^{s-1} z_0 = 0) \text{ の cusp のまわりの情報に帰着} \right)$ Wall, Arf inv. cobordism

③ ~~Atiyah-Singer~~ (index theory)

X spin $\Rightarrow$ ~~$D: \Gamma(S^+) \to \Gamma(S^-)$ Dirac~~ Sp_1-equivariant

~~$\mathrm{ind}\, D = -\frac{\sigma(X)}{4}$~~

$$\mathrm{ind}\, D = -\frac{\sigma(X)}{4}$$

D は $\mathbb{H}(Sp_1)$-equivariant より

$4 \mid \dim \ker D$, $4 \mid \dim \operatorname{coker} D$.

$\therefore\ 16 \mid \sigma(X)$ //

Kervaire-Milnor

$$P.D.\ \xi \underset{2}{\equiv} w_2(X) \qquad \left(\begin{array}{c} X\text{はspinで} \\ \text{なくともよい} \end{array}\right)$$

$$g_X(\xi) = 0$$

$$\Rightarrow \xi^2 \underset{16}{\equiv} \sigma(X)$$

∵ ① $\xi = [S^2 \hookrightarrow X]$ とせよ.

準備 blow up

$\xi^2 \geqq 0$ としてよい ($\xi^2 < 0$ ならXの向きを変えよ)

② $\tilde{X} := X \# (\xi^2+1)\overline{\mathbb{C}P^2}$

~~この操作をblow upと呼ぶ~~

$$\tilde{S} := S \# \cancel{(\xi^2+1)\overline{\mathbb{C}P^1}}\ \overline{\mathbb{C}P^1_1} \# \cdots \# \overline{\mathbb{C}P^1_{\xi^2+1}} \cong S^2$$

$$[\tilde{S}] = [S] + \cancel{(\xi^2+1)}[\overline{\mathbb{C}P^1_1}] + \cdots + [\overline{\mathbb{C}P^1_{\xi^2+1}}]$$

$$[\tilde{S}]^2 = [S]^2 - (\xi^2+1) = -1$$

blow down

③ $(\tilde{X}, \tilde{S}) \cong (X', \emptyset) \# (\overline{\mathbb{C}P^2}, \overline{\mathbb{C}P^1})$

obstruction to be spin

X' : spin

(応用)

適用 ― ④ Rohlin より $\sigma(X') \underset{16}{\equiv} 0$

$\Updownarrow$

計算 ― ⑤ 計算(ほとんど小学生) $\xi^2 \underset{16}{\equiv} \sigma(X)$ //

Cor. $g_{\mathbb{C}P^2}(3[\mathbb{C}P^1]) \geq 1$, etc. $\left(\begin{array}{c} \xi^2 = 9 \\ \sigma = 1 \end{array} \updownarrow 8\right)$

$3[\mathbb{C}P^1] = -P.D.\ K_{\mathbb{C}P^2}$

g_X と ~~…~~ Atiyah-Singer
"局所性"
(G-signature theorem)

§3 Atiyah-Singer と g_X (1970's)

~~Th (Hsiang-Szczarba,~~ Rohlin) (simple version)

Rohlin' $2 \mid \xi \Rightarrow g_X(\xi) \geq \left| \frac{1}{4}\xi^2 - \frac{1}{2}\sigma(X) \right| - \frac{1}{2} b_2(X)$

(証)

準備
① $\xi = [\Sigma_g \hookrightarrow X]$ とせよ.
② $\mathbb{Z}_2 \curvearrowright \exists \tilde{X} \xrightarrow{\mathbb{Z}_2} X$ branched along Σ_g
③ $\mathbb{Z}_2 \curvearrowright (H_2(\tilde{X}), g_{\tilde{X}}) \xmapsto{\otimes \mathbb{C}} (H_2(\tilde{X};\mathbb{C}), \tilde{g}_{\tilde{X}}) \curvearrowleft \mathbb{Z}_2$
sym. bilinear　　　hermitian
$\parallel$
$\mathbb{Z}_2 \curvearrowright$ の固有空間分解. $E_0 \oplus E_1$

適用 - ④ G-signature theorem より $\sigma(E_1) = \sigma(X) - \frac{\xi^2}{2}$
fixed pt set Σ_g のまわりのデータが必要

計算 - ⑤ $|\sigma(E_1)| \leq \dim(E_1) \underset{\text{代数的}}{=} b_2(X) + 2\,\text{genus}(\Sigma)$

より 与式. //

Cor. $g_{\mathbb{C}P^2}(2m[\mathbb{C}P^1]) \geq m^2 - 1 \fallingdotseq$ (conj.) の半分.

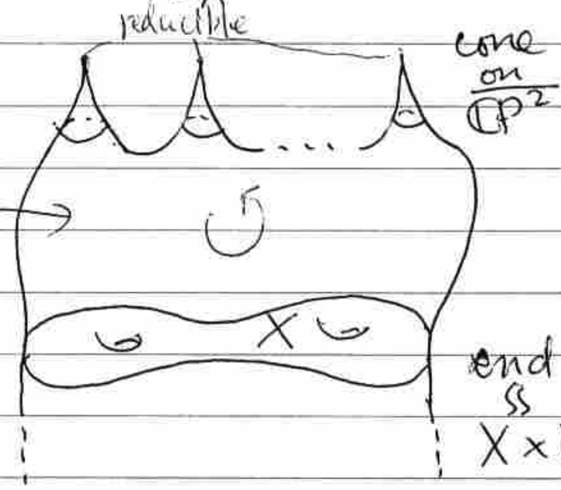
$$\left(\underset{\text{conj.}}{=} \frac{(2|m|-1)(2|m|-2)}{2} \right)$$

Hirai (1989) $X = \#2 S^2 \times S^2$ or $\#3 S^2 \times S^2$ etc.
(Hirano)
$g_X(\xi) = 0 \Leftrightarrow \xi^2 = 0$ or ξ primitive
(G-signature theorem $\Rightarrow$ '$\Rightarrow$')

§4 Donaldson & g_X (1980s ~)

<u>Donaldson A</u>　$b_2^+(X)=0 \Rightarrow g_X \cong (-1)^{\oplus b_2^-(X)}$

$k=-c_2=1$
ASD connection の
moduli space
dim $= 5$
⇑
index theorem



<u>Kuga</u>　$g_{S^2\times S^2}\left(p[S^2\times\cdot]+q[\cdot\times S^2]\right)=0 \Leftrightarrow \begin{cases}|p|\le 1\\ \text{or}\\ |q|\le 1\end{cases}$

∵ Kervaire-Milnor とほぼ平行.

準備
① $\xi=[S^2\hookrightarrow X]$ とせよ.　$\xi^2>0$ とせよ.
② blow up (ξ^2-1) pts
③ blow down $(\mathbb{CP}^2, \mathbb{CP}^1)$

適用 - ④ Donaldson A を使う (Rohlin でなく)

計算 - ⑤ arithmetic.　//

<u>K</u>　$X=\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$, $n\le 9$.
$\xi^2>0$
$\Rightarrow$ ∃判定法 $g_X(\xi)=0$ or >0

∵ ①~④はほぼ平行.
⑤ 双曲キカ.
⑥ <u>Wall</u>　$\mathrm{Diff}^+(X)\to \mathrm{Aut}(H_2(X);g_X)$ <u>surj.</u>

注) その後 Donaldson theory とともに $g_X(\xi)$ の

g_X と Seiberg-Witten

§5 Seiberg-Witten と g_X (1994〜)

Kronheimer-Mrowka $\quad g_{\mathbb{C}P^2}(d[\mathbb{C}P^1]) = \dfrac{(|d|-1)(|d|-2)}{2}$

(Thom conj. (original) の解決)

$|d| \geq 3$ $\left(\begin{array}{l} d=0 \\ d=\pm1,\pm2 \end{array}\right.$ は別に考える

(证) ① $\xi = [\Sigma^2 \hookrightarrow X]$, genus$(\Sigma) \geq 1$ とせよ.

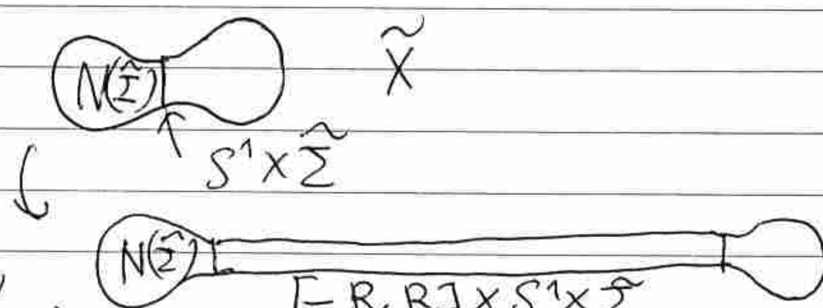
準備 ② $(\tilde{X}, \tilde{\Sigma}) := (X, \Sigma) \# \xi^2 (\overline{\mathbb{C}P^2}, \overline{\mathbb{C}P^1})$

blow up

③ $\tilde{X}$ の Riemann 計量を次のように変形.

- $\partial N(\tilde{\Sigma}) = S^1 \times \tilde{\Sigma}$ product metric
- $\cdot \times \tilde{\Sigma}$ 上 constant scalar curvature $-2\pi(4\,\text{genus}(\tilde{\Sigma}) - 4)$

適用 ④



$|d| > 3$ より $R \gg 1$ で $\exists$ SW 解.

計算 ⑤ $\boxed{}$ 上での SW 解の評価より

genus$(\Sigma) \geq$ (右)

//

(注) ⑤ は Kervaire-Milnor から Kronheimer-Mrowka へとだんだん難しく, 表面的な応用では

うまくいかなくなってきた

ユーザからスーパーユーザへ

同様にして(?)

Th (~~Ruberman~~, Li-Li, Tokui, Ue, K-, ...)

Li-Li (他)

$$g_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}(a,b) = \max\left\{0, \frac{(||a|-|b||-1)(|a|+|b|-2)}{2}\right\}$$

$$g_{S^2 \times \Sigma_g}(p,q) = \max\{0, (|p|-1)(|q|-1)\} + |q| g$$

$g_{X}(\xi)$ も計算可能 $(\forall \xi)$ $\begin{matrix} X \\ \downarrow S^2 \\ \Sigma_{g \geq 1} \end{matrix}$

Th (Ozváth-Szabó)

Ozváth-Szabó X symplectic / kähler, $\Sigma \subset X$ symplectic / hol. (symplectic Thom conj. の解決)

$\Longrightarrow$ $g_X([\Sigma]) = \text{genus}(\Sigma)$

とくに X kähler, Σ hol. でもよい.

また, ~~Hsiang-Szczarba~~ Rohlin + Furuta's $\frac{10}{8}$ より,

X spin, $\sigma \neq 0 \Rightarrow \frac{10}{8}|\sigma| + 2 \leq b_2$

Bryan $b_2^+(X) > 1$, $2 | \xi$, $\xi/2 \equiv_2 w_2(X)$

$$\Longrightarrow g_X(\xi) \geq \frac{10}{8}\left(\frac{\xi^2}{4} - \sigma(X)\right) - b_2(X) + 2$$

とくに $g_{\mathbb{CP}^2 \# \mathbb{CP}^2}(6[\mathbb{CP}^1] + 2[\mathbb{CP}^1]) = 10$

etc.

§6 g_X の現状

- map としての g_X が完全に計算された X は
$(S^4)\ \mathbb{C}P^2$, と $\begin{matrix} X \\ \downarrow S^2 \\ \Sigma_g \end{matrix}$ のみ.

(K3 も?)

- Thom conj が解決されても
一般には g_X は完全には計算 ~~不能~~ 可能でない.

$\mathrm{Im}\left(\mathrm{Diff}^+(X) \to \mathrm{Aut}(H_2(X), g_X)\right) \cdot \xi$
などを決定する必要がある.
(D理論, SW理論が有効)

◎ $g_{\mathbb{C}P^2 \# \mathbb{C}P^2}(3,2) = 1$? still open

X : not symplectic ~~symplectic~~ でない
$(\Leftarrow b_2^+(X) \underset{(2)}{\equiv} 0)$
ξ : not dual to $w_2(X)$ ξ ~~characteristic~~ primitive not dual to $w_2(X)$

Ind も D も SW も ✗

~~R~~ ~~Ind~~ ~~D~~ ~~SW~~