2001.02.19 質問

・spin structure　　　Dirac operator

・(Riemann 面に対する) Riemann-Roch の定理.

<u>Riemann 面</u>

X　2dim manifold

$X=\bigcup\limits_{\alpha} U_\alpha$ (open)　　　$z_\alpha : U_\alpha \hookrightarrow \mathbb{C}$

$$\begin{tikzcd}
 & U_\alpha \arrow[r, "z_\alpha"] & \mathbb{C} & \supset & z_\alpha(U_\alpha\cap U_\beta) \arrow[d, "f_{\beta\alpha}"] \\
U_\alpha\cap U_\beta \arrow[ur, hook] \arrow[dr, hook] & & \circlearrowleft & & \\
 & U_\beta \arrow[r, "z_\beta"'] & \mathbb{C} & \supset & z_\beta(U_\alpha\cap U_\beta)
\end{tikzcd}$$

$\forall f_{\beta\alpha}$: 正則

このとき X には <u>1 dim complex manifold</u> の str. が入っているという。
(R 面)

$z=x+yi$

$f: U \to \mathbb{C}$　smooth, $U \subset \mathbb{C}$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \underbrace{\boxed{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)}}_{=\frac{\partial f}{\partial z}}\underbrace{(dx+idy)}_{=dz} + \underbrace{\boxed{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}}_{=\frac{\partial f}{\partial \bar z}}\underbrace{(dx-idy)}_{=d\bar z}$$

$$= \partial f + \bar\partial f.$$

where $\partial f = \frac{\partial f}{\partial z}dz$, $\bar\partial f = \frac{\partial f}{\partial \bar z}d\bar z$

$U \subset \mathbb{C}$

$\bar\partial : \Gamma(U;\mathbb{C}) \longrightarrow$ "$\Gamma(U;\mathbb{C})\,d\bar z$"

$\{\mathbb{C}$値 なめらかな関数$\}$

$X = \bigcup U_\alpha \qquad z_\alpha : U_\alpha \hookrightarrow \mathbb{C}$ R面

$\bar\partial \quad \Gamma(X;\mathbb{C}) \longrightarrow \Gamma(X, \overline{T^*_{\mathbb{C}}X})$

{X上の ℂ値なめらか関数}

① TX : Xの接束

② XがR面の時、TXの各fiberはℂ上1次元のベクトル空間をもつ

そう思う時 TX のことを $T_{\mathbb{C}}X$ とかく。

③ $T^*_{\mathbb{C}}X \quad \underbrace{T_{\mathbb{C}}X \text{ の dual}} \quad \mathrm{Hom}_{\mathbb{C}}(T_{\mathbb{C}}X, \mathbb{C})$

④ $\overline{T^*_{\mathbb{C}}X}$

$T^*_{\mathbb{C}}X$ の複素共役

元のところは同じ.

$$\begin{array}{ccc} \mathbb{C} \times T^*_{\mathbb{C}}X & \longrightarrow & T^*_{\mathbb{C}}X \\ \downarrow \text{共役} \quad \downarrow \text{'id'} & \circlearrowleft & \downarrow \\ \mathbb{C} \times \overline{T^*_{\mathbb{C}}X} & \longrightarrow & \overline{T^*_{\mathbb{C}}X} \end{array}$$

演習問題

$X \supset U \xrightarrow{z} \mathbb{C}$

確認 $\Gamma(U, \overline{T^*_{\mathbb{C}}X}) = “\Gamma(U, \mathbb{C}) \cdot \overline{dz}”$

$$U_\alpha \cap U_\beta \begin{array}{l} \subset U_\alpha \xrightarrow{z_\alpha} \mathbb{C} \\ \subset U_\beta \xrightarrow[z_\beta]{} \mathbb{C} \end{array}$$

$$\Gamma(U_\alpha \cap U_\beta, \overline{T^*_{\mathbb{C}}X}) \begin{array}{l} \subset U_\alpha \quad \Gamma(U_\alpha \cap U_\beta, \mathbb{C}) \cdot \overline{dz_\alpha} \ni f\,\overline{dz_\alpha} \\ \subset U_\beta \quad f\,\overline{dz_\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathbb{C}) \cdot \overline{dz_\beta} \end{array}$$

※ $f \cdot \overline{dz_\alpha}$ "$=$" $g\,\overline{dz_\beta}$ $\iff$ $f = g\left(\overline{\dfrac{\partial z_\beta}{\partial z_\alpha}}\right)$

$z_\beta = z_\beta(z_\alpha)$ on $U_\alpha \cap U_\beta$. ⌟

$$\bar\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

⊛ $T_{\mathbb{C}}X$ に Hermite 内積を固定.

$$\begin{array}{ccccc} (T_{\mathbb{C}}X)_x & \times & (\overline{T_{\mathbb{C}}X})_x & \longrightarrow & \mathbb{C} \\ \Psi & & \Psi & & \Psi \\ u & x \in X & v & & (u, v) \end{array}$$

$u, v \in T_{\mathbb{C}}X$

$\overline{T^*_{\mathbb{C}}X}$ にも Hermite 内積が定まる.

$\bar\partial$ の formal adjoint $\bar\partial^* : \Gamma(X, \overline{T^*_{\mathbb{C}}X}) \longrightarrow \Gamma(X, \mathbb{C})$.

を def できる.

「 $\forall s \in \Gamma(X, \mathbb{C})$ compact support
$\forall t \in \Gamma(X, \overline{T^*_{\mathbb{C}}X})$

$$\int_X (\bar\partial s, t)\, \underset{\uparrow}{dvol} = \int_X (s, \bar\partial^* t)\, dvol.$$

volume form. ⌟

$\mathbb{C}$ 上

$f \overset{\bar\partial}{\longmapsto} \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\overline{dz}$

$\left(\frac{\partial}{\partial x}\right)^* = -\frac{\partial}{\partial x}$. 部分積分

$\left(\frac{\partial}{\partial y}\right)^* = -\frac{\partial}{\partial y}$.

$(i\text{倍})^* = -i\text{倍}$.

$g\overline{dz} \overset{\bar\partial^*}{\longmapsto} \frac{1}{2}\left\{-\frac{\partial g}{\partial x} + (-i)\left(-\frac{\partial g}{\partial y}\right)\right\}$

$$= -\frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right)$$

$$= -\frac{\partial g}{\partial z}.$$

$\underline{X : \text{compact}}$ と仮定

$$\Gamma(X, \mathbb{C}) \underset{\bar{\partial}^*}{\overset{\bar{\partial}}{\rightleftarrows}} \Gamma(X, \overline{T^*_{\mathbb{C}}X})$$

<u>Def</u>

$$\mathrm{ind} = \dim \operatorname{Ker} \bar{\partial} - \dim \underbrace{\operatorname{Ker} \bar{\partial}^*}_{\substack{\shortparallel \\ \operatorname{Coker} \bar{\partial}}} \quad \text{Fact.}$$

<u>拡張</u>

$$\begin{array}{l} L \\ \downarrow \quad \text{正則複素直線束.} \\ X \end{array}$$

$$\begin{array}{l} L = \bigcup U_\alpha \times \mathbb{C} \\ \downarrow \\ X = \bigcup U_\alpha \end{array}$$

$$\begin{array}{ccccc} U_\alpha \times \mathbb{C} & \supset & (U_\alpha \cap U_\beta) \times \mathbb{C} & & (p, u) \\ & & \downarrow & & \downarrow \\ U_\beta \times \mathbb{C} & \supset & (U_\alpha \cap U_\beta) \times \mathbb{C} & & (p, g_{\beta\alpha}(p) u) \end{array}$$

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^* \quad \text{正則.}$$

つまり. $\bar{\partial} g_{\beta\alpha} = 0$.

このとき.

$$\bar{\partial} : \Gamma(X, L) \to \Gamma(X, \overline{T^*_{\mathbb{C}}X} \otimes L)$$

を拡張して定義できる.

L に Hermite 内積 を ひとつ固定.

$$\Gamma(X, L) \underset{\bar\partial_L^*}{\overset{\bar\partial_L}{\rightleftarrows}} \Gamma(X, \overline{T^*_{\mathbb{C}}X} \otimes L)$$

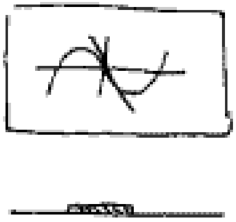
<u>Def</u>

$$\operatorname{ind}(L) = \dim \ker \bar\partial_L - \dim \underbrace{\ker \bar\partial_L^*}_{\underset{\operatorname{Coker} \bar\partial_L}{\shortparallel}\ \text{Fact}}$$

<u>R-R</u>

$$\operatorname{ind}(L) = \deg L + 1 - g(X)$$

$$\deg L = c_1(L)[X]$$



$$\begin{array}{c} L \\ \downarrow \uparrow s \\ X \end{array}$$

正則でなくてもよい.

generic なめらかな section

$\forall x \in s^{-1}(0)$ $\quad ds : (TX)_x \xrightarrow{\cong} (L)_x$

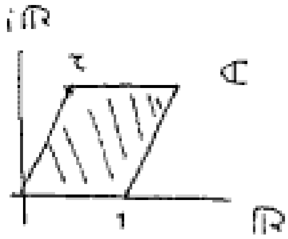
向きをたもつ $+1$ 　向きを保たない -1.

$s^{-1}(0)$ を 向きつき でかぞえる $\longrightarrow \deg L$

$$1 - \underset{\substack{\uparrow \\ X\text{の genus}}}{\underline{g(X)}} = \frac{1}{2} \underset{\substack{\uparrow \\ X\text{の Euler 数}}}{\underline{\chi(X)}} = \frac{1}{2} \deg T_{\mathbb{C}}X = -\frac{1}{2} \deg T^*_{\mathbb{C}}X$$

<u>例4</u> $X = \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau$

$\mathrm{Im}\,\tau > 0$



$T_{\mathbb{C}} X = X \times \mathbb{C}$

$T^*_{\mathbb{C}} X = X \times \mathbb{C}$
$\quad = (\mathbb{C} \times \mathbb{C}) / \mathbb{Z} + \mathbb{Z}\tau$

$\mathbb{C} \times \mathbb{C}$
$\quad \Downarrow$
$(z, u) \sim (z + a + b\tau, u)$
$\quad a, b \in \mathbb{Z}.$

Xとの 4つの cpx line bundle 正則 Hermite

$$\left\{\begin{array}{ll} L_1 = \mathbb{C} \times \mathbb{C} / \sim_1 & (z, v) \sim_1 (z + a + b\tau, v) \\ L_2 = \mathbb{C} \times \mathbb{C} / \sim_2 & (z, v) \sim_2 (z + a + b\tau, (-1)^a v) \\ L_3 = \mathbb{C} \times \mathbb{C} / \sim_3 & (z, v) \sim_3 (z + a + b\tau, (-1)^b v) \\ L_4 = \mathbb{C} \times \mathbb{C} / \sim_4 & (z, v) \sim_4 (z + a + b\tau, (-1)^{a+b} v) \end{array}\right.$$

$\mathrm{ind}(L_k)$ $k = 1, 2, 3, 4$ の直接計算

$\mathrm{ind}(L_1) = \dim \mathrm{Ker}\, \bar{\partial}_{L_1} - \dim \mathrm{Ker}\, \bar{\partial}^*_{L_1}$

$\bar{\partial}_{L_1} : f(z) \longmapsto \bar{\partial} f(z) = 0$
fが正則.

$f(z + a + b\tau) = f(z).$

$\mathrm{Ker}\, \bar{\partial}_{L_1} =$ 定数関数

$$\ker \bar\partial^*_{L_1} = \{ \text{定数(実)倍}\ d\bar{z} \}$$

$$\shortparallel$$

$$\sim \partial$$

$\bar{g}$ が正則,

$$\bar\partial^*_{L_1} \quad g(z)\, d\bar{z} \longmapsto -\frac{\partial g(z)}{\partial z} = 0$$

$$g(z + a + b\tau) = g(z)$$

$$\dim \ker \bar\partial_{L_1} = \dim \ker \bar\partial^*_{L_1} = 1$$

$$\therefore \ \operatorname{ind}(L_1) = 1 - 1 = 0. \quad \lrcorner$$

L_2 について (同)様な計算をすると,

$$\left.\begin{aligned} \bar\partial_{L_2} : \ f(z) &\longmapsto \bar\partial f(z) = 0 \\ f(z + a + b\tau) &= (-1)^a f(z) \end{aligned}\right\} \Rightarrow f = 0.$$

$\ker \bar\partial_{L_2} = 0$ (同)様に $\ker \bar\partial^*_{L_2} = 0$

$$\operatorname{ind}(L_2) = 0 - 0 = 0.$$

(同)様に, $\operatorname{ind}(L_k) = 0 \qquad k = 2, 3, 4$

$$\left(\because \ \ker \bar\partial^*_{L_k} = \ker \bar\partial^*_{L_k} = 0 \right)$$

主張

$$L = \text{“}\sqrt{T^*_{\mathbb{C}} X}\text{”}$$

のとき. (正確には, $L^2 = T^*_{\mathbb{C}} X$ のこと.)

$$\operatorname{ind} L = 0.$$

R.R.

$$\mathrm{ind}(L) = \deg L \underbrace{-\ \tfrac{1}{2}\deg T_{\mathbb{C}}^{*}X}_{1-g}$$

$$\left[\ L^2 = T_{\mathbb{C}}^{*}X \quad \Longrightarrow \quad 2\deg L = \deg T_{\mathbb{C}}^{*}X\ \right]$$

$\Longrightarrow$ RRは $\mathrm{ind}L=0$ を意味する.

トーラスの場合. L_k ($k=1,2,3,4$) L_k^2 $f(z+a+b\tau)=\pm f(z)$

$f^2(z+a+b\tau)=f^2(z)$.

つまり. $L_k^2 = T_{\mathbb{C}}^{*}X$

$\uparrow$

X トーラスなら自明.

$\mathbb{C}$ 上で $\left(\frac{\partial}{\partial \bar z}\right)^{*} = -\frac{\partial}{\partial z}$ であった.

これを一般の R面上に拡張したい. 仮定 $L^{\otimes 2} = T_{\mathbb{C}}^{*}X$

$T_{\mathbb{C}}^{*}X$ の Hermite 内積をいれた. $\Longrightarrow$ L に Hermite 内積が入る.

$$\Gamma(X, L) \xrightarrow{\bar\partial_L} \Gamma(X, \overline{T_{\mathbb{C}}^{*}X} \otimes L)$$

Hermite 内積 $L \otimes \bar L \xrightarrow{\cong} \mathbb{C}$.

$\Longrightarrow \bar L \cong L^{*}$

$T_{\mathbb{C}}^{*}X$ の Hermite 内積

$$\Longrightarrow \overline{T_{\mathbb{C}}^{*}X} = (T_{\mathbb{C}}^{*}X)^{*} \left(= T_{\mathbb{C}}X\right)$$

$$\overline{T_{\mathbb{C}}^{*}X} = (T_{\mathbb{C}}^{*}X)^{*} = (L^{2})^{*}$$

$$\begin{aligned}\overline{T_{\mathbb{C}}^{*}L} \otimes L &= (L^{2})^{*} \otimes L \\ &\cong L^{*} \\ &= \overline{L}.\end{aligned}$$

まとめ

$$\begin{array}{ccc} \Gamma(X, L) & \xrightarrow{\bar{\partial}_L} & \Gamma(X, \overline{L}) \\ \mathrm{id} \Big\| & \wr & \Big\| \mathrm{id} \\ \Gamma(X, \overline{L}) & \xrightarrow{(\overline{\bar{\partial}_L})} & \Gamma(X, L) \end{array}$$

formal adjoint $\bar{\partial}_L^{*}$

$$\Gamma(X, \overline{L}) \to \Gamma(X, L)$$

計算 $\Rightarrow$ $\bar{\partial}_L^{*} = -(\overline{\bar{\partial}_L})$

Cor

$$\operatorname{Ker} \bar{\partial}_L^{*} \underset{\substack{\uparrow \\ \text{ant. liner}}}{\cong} \operatorname{ker} \bar{\partial}_L$$

$$\dim \operatorname{Ker} \bar{\partial}_L^{*} = \dim \operatorname{ker} \bar{\partial}_L$$

$$\therefore \ \operatorname{ind} L = 0 \quad /\!/$$

Rem Fact

X の Riem metric $\Longrightarrow$ X上の Riemann 面としての コウゾウがきまる.
(TX の Euclid 内積)

$\underbrace{\dim \operatorname{Ker} \bar{\partial}_L}_{\substack{\| \\ H^{0}(L)}}$ $\underbrace{\dim \operatorname{Ker} \bar{\partial}_L^{*}}_{\substack{\| \\ \operatorname{Coker} \bar{\partial}_L = H^{1}(L)}}$ は X の R面 / L の 正則コウゾウ のみできまる.

<u>Rem</u> (つづき)

ind(L) は metric on X, L によらない.
cpx str on X
hol str on L によらない.

$L^{\otimes 2} = T^*_{\mathbb{C}} X$ のとき.

$$\dim \ker \bar{\partial}_L \mod 2 \in \mathbb{Z}/2\mathbb{Z}.$$

は metric によらない.

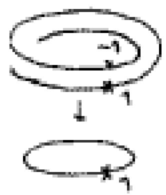
· <u>Spin str.</u>

X　2 dim Riem mfd <u>oriented</u>

$$\begin{array}{l} Fr \\ \downarrow SO(2) \\ X \end{array} \quad \text{oriented frame bundle} \qquad (Fr)_x = \left\{ (e_1, e_2) \;\middle|\; \begin{array}{l} e_1, e_2 \in (TX)_x \\ \text{正規直交キテイ} \\ \text{向きが指定} \\ \text{されたもの全体} \end{array} \right\}$$

$$(TX)_x = (Fr)_x \times_{SO(2)} \mathbb{R}^2$$

$$\text{2重被覆} \quad \begin{array}{c} Spin(2) \\ \downarrow \\ SO(2) \end{array}$$



$$\begin{array}{c} U(1) \ni w \\ \updownarrow \\ U(1) \ni w^2 \end{array}$$

<u>Def</u>

$$\begin{array}{ccc} \widetilde{Fr} & \longrightarrow & Fr \\ \downarrow Spin(2) & & \downarrow SO(2) \\ X & = & X \end{array}$$

X の spin structure とは.

① $$\begin{array}{c} \tilde{F}_r \\ \downarrow \ Spin(2) \\ X \end{array}$$

② $$\begin{tikzcd} \tilde{F}_r/\{\pm 1\} \arrow[r, "\varphi", "\cong"'] \arrow[d, "SO(2)"] & F_r \arrow[d, "SO(2)"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

<u>$\tilde{F}_r$ と φ のペア</u> のこと.

また $TX = F_r \times_{SO(2)} \mathbb{R}^2$

$SO(2) \curvearrowright \mathbb{R}^2$

<u>Def</u>

$$S^+ = \tilde{F}_r \times_{Spin(2)} \Delta^+$$

$$S^- = \tilde{F}_r \times_{Spin(2)} \Delta^-$$

$$\begin{array}{ccccccc} U(1) \ni w & & & \overline{\mathbb{C}} & \bar{a} & & w\text{倍} \\ \parallel & & & \parallel & \uparrow & & \\ Spin(2) & \curvearrowright & \Delta^+ = & \mathbb{C} & a & & w^{-1}\text{倍} \end{array}$$

$$Spin(2) \curvearrowright \Delta^- = \mathbb{C} \qquad w\text{倍}.$$

によって def.

<u>Rem</u>

$$\overline{S^+} \cong S^-$$

<u>2次元</u> (<u>$8k+2$</u> 次元)

のときに成立.

X: 3dim oriented Riem metric

$$\begin{array}{ccc} Fr & \xleftarrow[2\text{重被覆}]{} & \widetilde{Fr} \\ \downarrow SO(3) & & \downarrow Spin(3) \\ X & = & X \end{array}$$

$\Uparrow$ X の spin str.

$$\begin{array}{c} Spin(3) \\ \downarrow 2\text{重被覆} \\ SO(3) \end{array}$$

$S = \widetilde{Fr} \times_{Spin(3)} \Delta$

$Spin(3) \curvearrowright \Delta$

（奇数次元では $S^{\pm}$ はない。偶数次元で $S = S^{+} \oplus S^{-}$ と分かれる）

$\mathbb{C} = \mathbb{R} \oplus \mathbb{R} i \qquad i^2 = -1$

$\mathbb{H} = \mathbb{R} \oplus \underbrace{\mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k}_{= \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3}$

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k, \quad jk = -kj = i \\ ki = -ik = j \end{cases}$$

非可換体

$Sp(1) = \{ q \in \mathbb{H} \mid |q| = 1 \} \cong S^3 \qquad \left(\begin{array}{l} |q|^2 = q\bar{q} \\ \overline{ab} = \bar{b}\bar{a} \end{array} \right)$

$Sp(1) \curvearrowright \operatorname{Im}\mathbb{H}$

$q \qquad a \longmapsto q a \bar{q}$

$\qquad\qquad\qquad q a q^{-1}$

$Sp(1)/\pm 1$

$Sp(1)/\pm 1 \xrightarrow{\text{isom}} SO(3)$ <u>Fact</u> これは同型!!

Cor

$$\begin{tikzcd} \mathrm{Spin}(3) \arrow[d] & = & Sp(1) = S^3 \arrow[d] \\ SO(3) & = & Sp(1)/\pm 1 \cong \mathbb{R}P^3 \end{tikzcd}$$

Def

$$\mathrm{Spin}(3) \curvearrowright \Delta = \mathbb{H}$$

$$\mathrm{Spin}(3) = Sp(1) \ni q, \qquad \mathbb{H} \ni c \longmapsto qc$$

右からの $\mathbb{H}$ への掛け算と
$\mathrm{Spin}(3)$ 作用は可換

Rem

Δ は $\mathbb{H}$ 上の vect space
$8k+3$ 次元で成立.

X 4 dim

$$\mathrm{Spin}(4) \curvearrowright \Delta^+, \Delta^-$$

$$S^\pm = \tilde{F}_r \times_{\mathrm{Spin}(4)} \Delta^\pm$$

$\mathbb{R}^4 = \mathbb{H}$

$$Sp(1) \times Sp(1) \curvearrowright \mathbb{H}$$

$$(q_+, q_-) \qquad a \longmapsto q_+ a q_-^{-1}$$

$$\frac{Sp(1) \times Sp(1)}{\{(1,1), (-1,-1)\}}$$

$$\frac{Sp(1) \times Sp(1)}{\{(1,1), (-1,-1)\}} \xrightarrow{\text{hom}} SO(4).$$

Fact これは同型 !!

<u>Cor</u>

$$\begin{array}{ccccc}
\mathrm{Spin}(4) & = & Sp(1)\times Sp(1) & \cong & S^3\times S^3 \\
\downarrow & & & & \downarrow \\
SO(4) & = & \dfrac{Sp(1)\times Sp(1)}{\{(1,1),\ (-1,-1)\}} & &
\end{array}$$

<u>Def</u>

$\mathbb{R}^4 \cong \mathbb{H}$

$$\mathrm{Spin}(4) \curvearrowright \Delta^+ = \mathbb{H}$$

$\mathrm{Spin}(4) = Sp(1)\times Sp(1) \ni (g_+,\ g_-)$

$\mathbb{H} \ni c_+ \longmapsto g_+ c_+$

$\Delta^- = \mathbb{H} \ni c_- \longmapsto g_- c_-$

<u>Rem</u>

Δ^+, Δ^- は $\mathbb{H}$ 上の vect. space

[illegible]

$$\begin{array}{ccc}
H_2(\mathbb{C}P^2) & = & \mathbb{Z} \\
\Downarrow & & \Downarrow \\
\left[\{[z_0:z_1:z_2] \mid z_0^3+z_1^3+z_2^3=0\}\right] & = & 3
\end{array}$$