2001.02.20

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G : Lie grp $\quad X$: C^∞-mfd.

$$G \curvearrowright X \iff G \times X \to X \quad C^\infty$$
$$1x = x$$
$$g(hx) = (gh)x$$

$$(\, G \subset \mathrm{Diff}(X) \,)$$

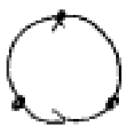
Smith Th $\quad G = \mathbb{Z}_p \quad p$ prime

$$X = \mathbb{Z}_p HS^d$$
$$\Longrightarrow$$
$$X^G = \mathbb{Z}_p HS \quad (\phi \text{もあり})$$

$\exists g$ gen.

$G \curvearrowright X \quad C^\infty$-closed mfd

G finite $= \mathbb{Z}_p$

$S^1 =$  $\mathbb{Z}_3 \circlearrowleft 120^\circ$

$X/G = X/\sim$

$X \ni x_0 \quad G(x_0)$: orbit

$C(K)$ chain cpx coef $\mathbb{Z}_p = \langle g \rangle$

$$\alpha : C(K) \to C(K)$$
$$\sigma \mapsto \sigma + g\sigma + \cdots + g^{p-1}\sigma$$

$$\beta : C(K) \to C(K)$$
$$\sigma \mapsto \sigma - g\sigma$$

$$(\alpha\beta = \beta\alpha = 0)$$

単体分割できる

$G \curvearrowright K$

$K \ni \sigma^k \xmapsto{g} g\sigma^k \in K$

実は K'' とっておけば

$X^G = |K''^G|$

$X/G = |K''/G|$

$G \curvearrowright X$ free

$\Rightarrow \chi(X) = |G|\,\chi(X/G)$

$G \curvearrowright X \setminus X^G$ free

$\Rightarrow \chi(X) = |G|\,\chi(X/G) - (|G|-1)\,\chi(X^G)$

$0 \to \alpha C(K) \oplus C(K^g) \to C(K) \xrightarrow{\rho} \rho C(K) \to 0$ exact

$0 \to \rho C(K) \oplus C(K^g) \to C(K) \xrightarrow{\alpha} \alpha(C(K)) \to 0$ exact

$\to H_{k+1}(\rho C(K)) \to H_k(\alpha C(K)) \oplus H_k(X^g) \to H_k(X) \to$

$\to H_{k+1}(\alpha C(K)) \to H_k(\rho(C)) \oplus H_k(X^g) \to H_k(X) \to$

$a_k = rk\, H_k(\alpha C(K))$

$b_k = \quad \rho$

$x_k = rk\, H_k(X)$

$y_k = rk\, H_k(X^g)$

$a_0 + y_0 \leqq b_1 + x_0$

$b_1 + y_1 \leqq a_2 + x_1$

$\vdots$

$a_n + y_n \leq \underset{\overset{\shortparallel}{0}}{b_{n+1}} + x_n$

$rk\, H_k(X^g) \leq rk\, H_k(X) = 2$

$\quad X \; \mathbb{Z}_p HS$

case 1 $rk\, H_k(X^g) = 0$ $\quad X^g = \phi$

case 2 rk 〃 $= 2$ $\quad X^g : \mathbb{Z}_p HS$

case 3 rk 〃 $= 1$ $\quad H_0(X^g) = \mathbb{Z}_p$ あとは0

$\chi(X^g) = 1$

$\mathbb{Z}_p \curvearrowright X$ 半自由 $\quad X \setminus X^g$.

$\underset{2}{\chi(X)} \equiv -\underset{-1}{\chi(X^g)}$ ムジュン //

1930s

1950s 同変コホモロジー（局所化）

$$H_G^*(X) = H^*(EG \times_G X)$$

$G \curvearrowright X$ free $X \xrightarrow{G} X/G$ $EG \times_G X \sim X/G$

$$H_G^*(X) = H^*(X/G)$$

$$\begin{array}{c} EG \times_G X \\ \downarrow \pi \\ BG \end{array} \qquad \pi^*: H^*(BG) \to H_G^*(X)$$

$H_G^*(X)$ は $H^*(BG)$-algebra

$G \curvearrowright X$ 自明 $\Rightarrow$ $H_G^*(X) = H^*(BG \times X)$

$$\underset{\text{coeff 体}}{=} H^*(BG) \otimes H^*(X)$$

S^1, $\mathbb{Z}_p$ (p prime ≥ 3) $\curvearrowright \mathbb{C}P^n$

$H_G^*(\mathbb{C}P^n; \mathbb{Z}) = H^*(BG) \otimes H^*(\mathbb{C}P^n)$ (module)

"$H_G^*(X)$ にちょっとさいく" $\underset{\text{局所化}}{=}$ "$H_G^*(X^G)$ に同じサイク"

さいくとは $G = S^1$ $\quad S^{-1}H_{S^1}^*(X) = S^{-1}H_{S^1}^*(X^{S^1})$

$$\begin{array}{c} S^\infty = ES^1 \\ \downarrow \\ \mathbb{C}P^\infty = BS^1 \end{array} \qquad H^*(BS^1) = \mathbb{Z}[c]$$

$c \in H^2(BS^1)$

$ES^1 \xrightarrow[S^1]{} BS^1$ の chern class

$S = \{1, nc^r \mid n, r \in \mathbb{N}\}$ 積で閉.

$\swarrow H^*(BS^1)$-module

$S^{-1}M = S \times M/\sim \quad \ni \frac{m}{s}$

$(s_1, m_1) \sim (s_2, m_2) \Leftrightarrow t s_1 m_2 = t s_2 m_1, \quad t \in S$

orbit $\quad G \curvearrowright X \quad G(x_0) = G/G_{x_0}$

$\uparrow$ orbit

$S^1 \supset \mathbb{Z}_m$

$$H^*(BS^1) = H^*_{S^1}(S^1/S^1) \to H^*_{S^1}(S^1/\mathbb{Z}_m)$$
$$mc \longmapsto 0$$

$\{x \mid S^1/\mathbb{Z}_m = G(x)\} = X_m \subset U_m$

$U_m \to X_m \to S^1/\mathbb{Z}_m$

$$H^*_{S^1}(S^1/\mathbb{Z}_m) \to H^*_{S^1}(U_m)$$

$H^*(BG) \nearrow \quad mc \nearrow 0 \longrightarrow 0$

$$\underline{S^{-1}H^*_{S^1}(U_m) = 0}$$

- $X^{S^1} \neq \phi \Leftrightarrow H^*(BS^1) \xhookrightarrow{\pi^*} H^*_{S^1}(X)$

($\Rightarrow$ S^1でなくても
$\Leftarrow$ れんしゅうもんだい) $ES^1 \times_{S^1} X \xrightarrow{\pi} BG$

$\Big(H^*_{S^1}(\mathbb{C}P^n; \mathbb{Z}) = H^*(BS^1) \otimes H^*(\mathbb{C}P^n)$

$\therefore \mathbb{C}P^{n\,S^1} \neq \phi \quad // \Big)$

指数定理の応用

$G \curvearrowright X$ $\quad G \curvearrowright TX$ $\quad g: X \to X$

$dg_x : T_x X \to T_{g(x)} X$

$X^G \ni x$ $\quad \underset{\text{線型}}{G \curvearrowright T_x X}$ $\quad G \curvearrowright V$ Gの表現

Smith $\quad \mathbb{Z}_p \curvearrowright X$ $\quad \mathbb{Z}_p HS$ $\quad X^{\mathbb{Z}_p} = \{P, Q\}$

$G \curvearrowright T_P X$ と $G \curvearrowright T_Q X$ は同じか？

「同じである.」 Smith 予想

指数定理 $\quad \mathrm{ind} : K(X) \to \mathbb{Z}$

$$[\sigma(D)] \in K(X)$$

$$\mathrm{ind}\, D = \mathrm{ind}[\sigma(D)]$$

$$\mathrm{ind}_G : K_G(TX) \to R(G)$$

$$\mathrm{ind}_G D \in R(G)$$

$K_G(*) = R(G)$

$TX \longrightarrow *$

$R(G) \to K_G(TX)$ により $K_G(TX)$ は $R(G)$-module

$G = \langle g \rangle$ と仮定.

$$R(G) \supset S = \{ \quad tr(g|V) \neq 0 \}$$

$$TX^g \overset{i}{\hookrightarrow} TX \qquad K_G(TX) \underset{i_!}{\overset{i^*}{\rightleftarrows}} K_G(TX^g)$$

$$X^g \subset N^g \subset TX \qquad S^{-1}K_G(TX) \underset{i_!}{\overset{i^*}{\rightleftarrows}} S^{-1}K_G(TX^g)$$

$i_!$ → 同型 $\cong$

$$\begin{tikzcd}
K_G(TX) \arrow[r, "ind_G"] \arrow[d, bend right, dashed, "i^*"'] & R(G) \\
K_G(TX^g) \arrow[u, "i_!"'] \arrow[ur, "ind_G"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{-1}K_G(TX) \arrow[r] \arrow[d, bend right, "\cong"'] & S^{-1}R(G) \\
S^{-1}K_G(TX) \arrow[u, "\cong"'] \arrow[ur] &
\end{tikzcd}$$

$$S^{-1}K_G(TX^g)$$
$$\quad \in$$
$$i^* i_!(x) = [\Lambda_{-1} N^g_{\mathbb{C}}]\, x$$

$$\begin{tikzcd}
TX \arrow[r] & X \\
TN = \pi^* N^g_{\mathbb{C}} \arrow[u, phantom, "\cup"] \arrow[d] & N^g \arrow[u, phantom, "\cup"] \arrow[d] \\
TX^g \arrow[r, "\pi"] & X^g
\end{tikzcd}$$

$$K(X^g) \xrightarrow{\pi^*} K(TX^g)$$

$$\mathrm{Ind}^X_G u = \mathrm{ind}^{X^g}_G \frac{i^* i_! (i_!^{-1}) u}{[\Lambda_{-1} N^g_{\mathbb{C}}]}$$

$$= \mathrm{ind}^{X^g}_G \frac{i^* u}{[\Lambda_{-1} N^g_{\mathbb{C}}]}$$

$$= (\mathrm{ind}^{X^g}_{x|_1}) \frac{i^* u}{[\Lambda_{-1} N^g_{\mathbb{C}}]}$$

$G \curvearrowright V \quad \chi(G) \xrightarrow{tr_g} \mathbb{C}$

$X^g = \{p_1, \cdots, p_r\} \qquad K_G(X^g) = \oplus R(G)$

$$(\mathrm{ind}^{X^g}_{a,d})\left(\frac{i^* u(g)}{[\Lambda_{-1} TX \otimes \mathbb{C}]}\right)(g) = \sum_{i=1}^{r} \frac{tr(g|u_p)}{tr_g[\Lambda_{-1} T_{p_i}X \otimes \mathbb{C}]}$$

$$\sum (-1)^k tr(A|\Lambda^k V) \underset{ex.}{=} \det(1-A)$$

$$= \sum (-1)^k tr(dg_p) = \det(1-dg_p)$$

例4 de Rham

$$\mathrm{index}(\mathrm{de\ Rham})(g) = \sum (-1)^k tr(g|H^k(X)) = L(g)$$

古典的Lefschetz数

分子 $tr(g|\Lambda_{-1} T_p^* X) = \cdots = \det(1-dg_p)$

$\mathrm{ind}_G(\mathrm{de\ Rham}) = \sum_{p \in X^g} 1 =$ 点の数.