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(G-) Signature Theorem

応用 $\Sigma \subset X^4$ genus の評価 Rocklin Hsiang-Szczarba

Dirac operator の index $\Rightarrow$ vanishing

V-manifold の index Th の計算例 (Kawasaki)

X^n $\dim X = n$ cl. or. sm. mfd $n = 4k$

$$H^{2k}(X, \mathbb{Z})/\mathrm{Tor} \times H^{2k}(X, \mathbb{Z})/\mathrm{Tor} \to \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto (\alpha \cup \beta)[X]$$

$H^{2k}_{\pm}$ = maximal posi (nega) definite subspace

$$\mathrm{Sign}\, X = \dim H^{2k}_{+} - \dim H^{2k}_{-}$$

X $\Lambda^p T^* X$ $\Omega^p = \Gamma(X, \Lambda^p T^* X)$ p-form

$$d : \Omega^p \to \Omega^{p+1}$$

Riemannian metric $\rightsquigarrow$ $* : \Omega^p \to \Omega^{n-p}$, $\varphi \mapsto *\varphi$

$$\varphi \wedge * \varphi = |\varphi|^2 d\mathrm{vol}$$

d の adjoint $d^* = (-1)^{np+n+1} * d * : \Omega^{p+1} \to \Omega^p$

Ω^pの内積 $\langle \varphi, \psi \rangle = \int_X \varphi \wedge *\psi$

$$\langle d\varphi, \psi \rangle = \langle \varphi, d^*\psi \rangle$$

$D = d + d^* : \Omega^* \otimes \mathbb{C} \leftrightarrow \Omega^* \otimes \mathbb{C}$

<u>involution</u> $(n = 2\ell)$ $\quad \tau = i^{p(p-1)+\ell} * : \Omega^p \to \Omega^{n-p}$ （$\otimes \mathbb{C}$を省略）

$\tau^2 = 1$

$\Omega^* = \underset{\tau=1}{\Omega^+} \oplus \underset{\tau=-1}{\Omega^-} \qquad \tau D = -D\tau$

$D^\pm = D|_{\Omega^\pm}$

$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : \begin{matrix} \Omega^+ \\ \oplus \\ \Omega^- \end{matrix} \qquad D^* = D \qquad (D^\pm)^* = D^\mp$

$\operatorname{index} D^+ = \dim \operatorname{Ker} D^+ - \dim \underset{\underset{\operatorname{Ker} D^-}{\parallel}}{\operatorname{Ker} (D^+)^*} \in \mathbb{Z}$

<u>$n = 4k$</u> $\quad \operatorname{Ker} D = \operatorname{Ker} D^2 \cong \underset{\circlearrowleft \tau}{H^*_{DR} \otimes \mathbb{C}} = \underset{\tau=1}{H^+} \oplus \underset{\tau=-1}{H^-}$

$\operatorname{Ker} D^+ = H^+ \cap H^{2k} \oplus \left(H^+ \cap \sum_{0 \le p < 2k} (H^p \oplus H^{n-p}) \right)_{x \oplus \tau x}$

$\operatorname{Ker} D^- = H^- \cap H^{2k} \oplus \left(H^- \cap \quad " \quad \right)_{x \oplus -\tau x}$

$\Rightarrow \operatorname{index} D^+ = \dim H^+ \cap H^{2k} - \dim H^- \cap H^{2k}$

$\left(\begin{matrix} \alpha \in H^\pm \neq 0 \\ \tau = * \end{matrix} \quad \int \alpha \wedge \bar{\alpha} = \int \alpha \wedge \pm * \bar{\alpha} = \pm \langle \alpha, \alpha \rangle \gtrless 0 \right)$

<u>index Th</u>

$$\mathrm{index}\, D^+ = (-1)^n \mathrm{ch}\, \sigma(D^+)\, \mathrm{td}(TX\otimes\mathbb{C})\,[TX]$$
$$= \frac{\mathrm{ch}(\Omega^+ - \Omega^-)}{e(TX)}\, \mathrm{td}(TX\otimes\mathbb{C})[X]$$

$E \to X$ cpx vec b

$c(E) = 1 + c_1(E) + \cdots = \prod_{i=1}^{s}(1+x_i)$ dim fib $= 2s$

$\left(E = L_1 \oplus \cdots \oplus L_s \text{ のとき} \quad c(L_i) = 1 + \underset{c_1(L_i)}{\underset{\shortparallel}{x_i}}\right)$

$\mathrm{ch}\, E = \sum e^{x_i} = \mathrm{rk}\, E + c_1 E + \cdots$

$\mathrm{td}\, E = \prod \frac{x_i}{1-e^{-x_i}} = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \cdots$

$e(E) = \prod x_i$ $\quad c(E) = \prod(1+x_i)$

となるようにとれる.

$TX \cong P_1 \oplus \cdots \oplus P_{2k}$ $\quad P_i$ ori. 2-plane b.

$\quad L_1 \quad\quad L_{2k}$ $\quad L_i$ as cpx line b.

$$(\Omega^+ - \Omega^-)(L_i\otimes\mathbb{C}\oplus L_j\otimes\mathbb{C}) \cong (\Omega^+-\Omega^-)(L_i\otimes\mathbb{C}) \oplus (\Omega^+-\Omega^-)(L_j\otimes\mathbb{C})$$

P_i : fiberの basis $e_1 \overset{}{\underset{J}{\curvearrowright}} e_2$ $\quad L_i\otimes\mathbb{C} = L_i \oplus \overline{L_i}$

L_i : $e_1 - Je_2$ $\quad \Omega^+(L_i\otimes\mathbb{C}) \cong 1 \oplus \overline{L_i}$

$\Omega^-(L_i\otimes\mathbb{C}) \cong 1 \oplus L_i$

$$\mathrm{ch}(\Omega^+ - \Omega^-(L_i \otimes \mathbb{C})) = \mathrm{ch}(\bar{L}_i - L_i) = e^{-x_i} - e^{x_i} \qquad x_i = c_1(L_i)$$

$$\text{右辺} = \prod^{2k} \frac{e^{-x_i} - e^{x_i}}{x_i} \, \frac{x_i}{1 - e^{-x_i}} \, \frac{-x_i}{1 - e^{x_i}} [X]$$

$$= \prod^{2k} \frac{x_i (e^{-x_i} - e^{x_i})}{(1 - e^{x_i})(1 - e^{-x_i})} [X]$$

$$= 2^{2k} \prod \frac{\frac{x_i}{2}}{\tanh \frac{x_i}{2}} [X]^{4k} = \underbrace{\prod \frac{x_i}{\tanh x_i}}_{x_i^2 \text{ の多項式}} [X] = L[X]$$

$\Downarrow$ Pontrjagin class p_i で書ける.

$$p_i(TX) = (-1)^i c_{2i}(TX \otimes \mathbb{C})$$

$$p = 1 + p_1 + \cdots \qquad L(X) = 1 + \frac{1}{3} p_1 + \cdots$$

$\dim X = 4$ のとき $\quad \mathrm{Sign}\, X = \frac{1}{3} p_1 [X]$ $\lrcorner$

G-signature Th $\quad G \curvearrowright X \qquad \varphi \in \Omega^p$

(G-inv metric を入れる) $\quad$ cpt Lie grp. $\qquad g \cdot \varphi = (g^{-1})^* \varphi$

$$\underset{\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}}{\overset{}{D}} = d + d^* : \underset{\Omega^+ \oplus \Omega^-}{\Omega^*} \to \Omega^* \quad G\text{-不変}$$

$$\mathrm{ind}\, D^+ = \mathrm{Ker}\, D^+ - \mathrm{Ker}\, D^- \in R(G)$$

$$g \in G \quad \underset{G\text{-signature}}{\mathrm{Sign}(g, X)} = \mathrm{tr}\, g|_{\mathrm{Ker}\, D^+} - \mathrm{tr}\, g|_{\mathrm{Ker}\, D^-}$$

$X^g = \{x \in X \mid gx = x\}$ Atiyah-Singer Fixed Pt Th.

$j : X^g \subset X$ $\quad TX|_{X^g} = TX^g \oplus N^g$

N^g: Normal b.

$$\mathrm{Sign}(g, X) = (-1)^{\underset{m}{\underset{\|}{\dim X^g}}} \frac{ch^g\, j^* \sigma(D^+)\ td(TX^g \otimes \mathbb{C})}{ch^g(\Lambda_{-1} N^g \otimes \mathbb{C})}[TX^g]$$

$$\Lambda_{-1} E = \sum (-1)^i \Lambda^i E.$$

$$ch_g(a \otimes \underset{g\text{-表現}}{V}) = ch(a) \cdot \mathrm{tr}\, \rho(g) \qquad K_{\langle g \rangle} TX \to H_c^*(TX)$$

$$\mathrm{Sign}(g, X) = (-1)^{\frac{m(m+1)}{2}} \frac{ch_g \cancel{(\Omega^+ - \Omega^-)|_{TX \otimes \mathbb{C}}} \ (\Lambda^+ - \Lambda^-)|_{TX^g \otimes \mathbb{C} + N^g \otimes \mathbb{C}}}{e(TX^g) \cdot ch_g \Lambda_{-1} N^g \otimes \mathbb{C}}$$

$$\cdot\, td(TX^g \otimes \mathbb{C})[X^g]$$

特別な場合

$X^g = 1\mathrm{pt}$

$N^g \cong \bigoplus N^g_\theta \qquad N^g_\theta \cong \sum L_\theta$

$g = e^{i\theta}$

$$ch^g(\Lambda^+ - \Lambda^-)(L_\theta \otimes \mathbb{C})$$

$$= ch^g(\overline{L_\theta^i} - L_\theta^i) = \cdots$$

$$\mathrm{Sign}(g, X) = \prod (-i) \cot\frac{\theta}{2}$$

$$- \cot\frac{\theta_1}{2} \cot\frac{\theta_2}{2}$$