M. Furuta II

§ 復習と補足

• 和公式

m次対称行列値

$$A_{-\infty} \quad A(x) \quad A_{+\infty} \overset{\text{仮定}}{=} B_{-\infty} \quad B(x) \quad B_{+\infty}$$

$$A(x) \underbrace{\qquad\qquad}_{l} B(x) = C(x) \quad (A \#_l B)$$

$$D_A = \begin{pmatrix} 0 & D_{0A}^* \\ D_{0A} & 0 \end{pmatrix} \qquad D_{0A} = \nabla + A \qquad \nabla = \frac{d}{dx}$$

$$D_{0A}^* = -\nabla + A$$

$$\operatorname{ind} D_A = \dim \operatorname{Ker}_0 D_{0A} - \dim \operatorname{Ker}_0 D_{0A}^*$$

b

$$\boxed{\operatorname{ind} D_C = \operatorname{ind} D_A + \operatorname{ind} D_B + \dim \operatorname{Ker} A_{+\infty}}$$

Ker_0 $\pm\infty$で0に収束する解

$\cap$

Ker_b $\pm\infty$で bounded な解 収束する

注 $A_{\pm\infty}$ 0固有値をもたなければ $\operatorname{Ker}_0 = \operatorname{Ker}_b$

$S^1\times[0,\infty)$ $S^1\times[-\infty,0]$

• 例 $X_0 =$  $= X_1$

$X_0 \#_l X_1$ 

l

$S^1\times[0,l]$

X_0, X_1 に上のような metric を入れる。 $\Longrightarrow$ これから X_0 X_1 は Riemann面になる

Riemannian

・ X_0 上の Dolbeault op $\quad \bar\partial_{X_0} : \Gamma(X_0, \mathbb{C}) \to \Gamma(X_0, \overline{T^*_{\mathbb{C}}X_0})$

$$\bar\partial^*_{X_0} : \Gamma(X_0, \overline{T^*_{\mathbb{C}}X_0}) \to \Gamma(X_0, \underset{T^*_{\mathbb{C}}X_0 \otimes \overline{T^*_{\mathbb{C}}X}}{\overset{\shortparallel}{\mathbb{C}}})$$

$T^*_{\mathbb{C}}X_0$ Hermite 内積

$\overline{T^*_{\mathbb{C}}X_0} = (T^*_{\mathbb{C}}X_0)^*$

$\underline{T^*_{\mathbb{C}}X_0 : \mathrm{hol}}$

$$\Big\| \qquad \Big\downarrow \text{cpx anti linear}$$

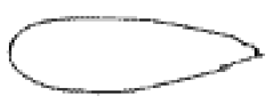
$$-\bar\partial_{X_0,K} : \Gamma(X_0, \underbrace{T^*_{\mathbb{C}}X_0}_{K^{\mathrm{hol}}}) \to \Gamma(X_0, T^*_{\mathbb{C}}X_0 \otimes \underbrace{T^*_{\mathbb{C}}X_0}_{K})$$

$$D_{X_0} = \begin{pmatrix} & \bar\partial^*_{X_0} \\ \bar\partial_{X_0} & \end{pmatrix}$$

$$\mathrm{ind}\, D_{X_0} = \dim \mathrm{Ker}_0\, \bar\partial_{X_0} - \underbrace{\dim \mathrm{Ker}_b\, \bar\partial^*_{X_0}}_{\dim \mathrm{Ker}_b\, \bar\partial_{X_0,K}}$$

$\mathrm{Ker}_0\, \bar\partial_{X_0} = \{ f : X \xrightarrow{\mathrm{hol}} \mathbb{C}$ s.t. ∞ で 0 に収束 $\} = \{0\}$

$S^1 \times [0, \infty) \ni e^{i\theta}, x \quad \xmapsto{\mathrm{hol}} \quad z = e^{-(x+i\theta)}$

$X_0 \cup \{\infty\} = \bar{X}_0$  $= S^2 = \mathbb{P}^1$

$\mathrm{Ker}_b\, \bar\partial_{X_0,K} = \{ \alpha :$ 正則1形式 s.t. $S^1 \times (0,\infty)$ 上で $\alpha = g\,(dx + i d\theta)$ と表すと $g \to$ 収束 $(x \to +\infty) \} = 0$

$\alpha = -g(z)\dfrac{dz}{z} \qquad x + i\theta = -\dfrac{dz}{z}$

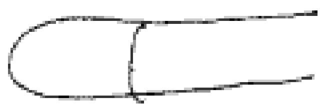
$z \to 0$ のとき $g(z)$ は有界 $\Longrightarrow$ $z=0$ で α は高々1位の極をもつ

 $\mathbb{P}^1$ 1点のみで高々1位の pole をもつ正則1形式はない

まとめ. $\mathrm{Ker}_0\, \bar\partial_{X_0} = 0 \qquad \mathrm{Ker}_0\, \bar\partial_{X_0}^{\,*} = 0$

$\mathrm{Ker}_b\, \bar\partial_{X_0} \cong \mathbb{C} \qquad \mathrm{Ker}_b\, \bar\partial_{X_0}^{\,*} = 0$

$\therefore\ \mathrm{ind} = 0 - 0 = 0$



$e^{i\theta}$ $\quad x$

$x + i\theta$

on $S^1 \times [0, \infty)$

$$\bar\partial_{X_0} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial \theta}\right)(dx + i\,d\theta)$$

これを使って $\overline{T_{\mathbb{C}}^* X}$ の自明化をとる.

$$2\bar\partial_{X_0} = \frac{\partial}{\partial x} + A \qquad A : C^\infty(S^1, \mathbb{C})\ で$$

$$A = i\frac{\partial}{\partial \theta} \qquad \underline{A^* = A}$$

X_0  X_1

$X_0 \#_\ell X_1$ $\qquad \mathbb{P}^1$

$$\underbrace{\mathrm{ind}\, D_{X_0 \# X_1}}_{\parallel} = \underbrace{\mathrm{ind}\, D_{X_0}}_{\parallel\ 0} + \underbrace{\mathrm{ind}\, D_{X_1}}_{\parallel\ 0} + \underbrace{\dim \mathrm{Ker}}_{\parallel\ 1}\ \underbrace{A_{+\infty}}_{\parallel\ i\frac{\partial}{\partial\theta}}$$

$$\dim \underbrace{\mathrm{Ker}\, \bar\partial_{\mathbb{P}^1}}_{\mathbb{C}} - \dim \underbrace{\mathrm{Ker}\, \bar\partial_{\mathbb{P}^1, K}}_{0}$$

$$1 \quad - \quad 0 \quad = 1$$

//

E_X^0, E_X^1 herm. cpx vect. b.

$\downarrow$

§ 積公式　X closed Riem. mfd

"楕円型" Dolbeault (Dirac)

$$D_X = \begin{pmatrix} 0 & D_{0X}^* \\ D_{0X} & 0 \end{pmatrix} : \begin{matrix} \Gamma(E_X^0) \\ \oplus \\ \Gamma(E_X^1) \end{matrix} \to \begin{matrix} \Gamma(E_X^0) \\ \oplus \\ \Gamma(E_X^1) \end{matrix}$$

$(Y, E_Y^0 \oplus E_Y^1, D_Y)$ も同様のデータとする。

$E_X = E_X^0 \oplus E_X^1$　ε_{E_X} についての固有分解

$\varepsilon_{E_X} \in \mathrm{End}\, E$　$\varepsilon_{E_X}^2 = \mathrm{id}_E$ | $\varepsilon_{E_Y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\| \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$\pi_X^* E_X \otimes \pi_Y^* E_Y = (\pi_X^* E_X^0 \otimes \pi_Y^* E_Y^0 \oplus E_X^1 \otimes E_Y^1) \oplus (E_X^0 \otimes E_Y^1 \oplus E_X^1 \otimes E_Y^0)$

$\tilde{\varepsilon} = \varepsilon_{E_X} \otimes \varepsilon_{E_Y}$

$(\pi_X^* E_X^0 \otimes \pi_Y^* E_Y^0 \oplus E_X^1 \otimes E_Y^1) = \tilde{E}^0$, $(E_X^0 \otimes E_Y^1 \oplus E_X^1 \otimes E_Y^0) = \tilde{E}^1$

$\tilde{D} : \Gamma(\tilde{E}) \to \Gamma(\tilde{E})$

$\| \quad \boxed{D_X \otimes \mathrm{id}_Y + \varepsilon_X \otimes D_Y}$

これは $\tilde{\varepsilon}$ と反可換　| D_X と ε_X は反可換 $\Leftrightarrow D_X = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$

$\tilde{\varepsilon} = \varepsilon_X \otimes \varepsilon_Y$　| D_Y と ε_Y 〃 $\Leftrightarrow D_Y =$ 〃

$\tilde{D}$: self-adjoint $\tilde{D}^* = \tilde{D}$

積公式　$\mathrm{ind}\, \tilde{D} = \mathrm{ind}\, D_X \, \mathrm{ind}\, D_Y$

(ii) $\ker \tilde{D} = \ker \tilde{D}_0 \oplus \ker \tilde{D}_0^*$

$\underline{s\text{-}\dim \ker \tilde{D}} := \dim \ker \tilde{D}_0 - \dim \ker \tilde{D}_0^*$

<u>claim</u> $\ker \tilde{D} \cong \ker D_X \otimes \ker D_Y$

$\left(\Rightarrow s\text{-}\dim \ker \tilde{D} = (s\text{-}\dim \ker D_X) \cdot (s\text{-}\dim \ker D_Y) \right)$

$\cdot \ \ker D_X = \ker D_X^2 \Leftarrow \left(\subset \text{明らか} \ \supset \ \boxed{\int_X (f, D_X^2 f) = \int_X \|D_X f\|^2} \right)$

$\ker D_Y = \ker D_Y^2$

$\ker \tilde{D} = \ker \tilde{D}^2$

$$\tilde{D}^2 = (D_X \otimes id_Y + \varepsilon_X \otimes D_Y)^2 = D_X^2 \otimes id_Y + id_X \otimes D_Y^2$$

$$\int_{X \times Y} (\tilde{f}, \tilde{D}^2 \tilde{f}) = \int_{X \times Y} (\tilde{f}, (D_X^2 \otimes id_Y + id_X \otimes D_Y^2) \tilde{f})$$

$$\int_Y \int_X (\tilde{f}, (D_X^2 \otimes id_Y) \tilde{f}) = \int_Y \int_X \|(D_X \otimes id_Y) \tilde{f}\|^2$$

$$\int_X \int_Y (\tilde{f}, (id_X \otimes D_Y^2) \tilde{f}) = \int_X \int_Y \|(id_X \otimes D_Y) \tilde{f}\|^2$$

$\Rightarrow \tilde{f} \in \ker D_X \otimes \ker D_Y$ claimの証明終 //

- ねじれた積のとき

$$\begin{array}{c} P \\ \downarrow G \text{ cpt Lie grp} \\ X \text{ closed Riem} \end{array}$$

$Y \circlearrowleft G$ が作用. G-inv. Riem. metric.

$\tilde{X} := P \times_G Y$ ねじれた積

$$\begin{array}{c} P \times_G Y \\ \downarrow Y \\ X \end{array}$$

$E_Y = E_Y^0 \oplus E_Y^1$

$\downarrow$ G 同変ベクトル束

Y

$D_Y = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ G+同変.

~~$E_X = E_X^0$~~

$E_X = E_X^0 \oplus E_X^1$

$\downarrow$

X

$D_X = \begin{pmatrix} 0 & D_{X_0}^* \\ D_{X_0} & 0 \end{pmatrix}$

仮定 $\ker D_Y \circlearrowleft G$ 作用は trivial と仮定.

$\tilde{E} = \boxed{\pi_X^* E_X} \otimes \boxed{P \times_G E}$

$$\tilde{X} \xrightarrow{\pi_X} X \qquad P \times_G E \to P \times_G Y = \tilde{X}$$

$\tilde{D}$ もさっきのコウセイをうまくねじって定義する.

積公式 $\quad \mathrm{ind}\, \tilde{D} = \mathrm{ind}\, D_X\, \mathrm{ind}\, D_Y$

実は $\quad \ker \tilde{D} \cong \ker D_X \otimes \ker D_Y$

$Y =$  $\circlearrowleft SO(2)$

$P \qquad \hat{X} = P \times_G Y \rightsquigarrow \tilde{E}$ と $\tilde{D}$ は上のように def できる

$\downarrow SO(2)$

X

$\begin{cases} \cdot\ D_X \text{ on } \Gamma(E_X) \\ \cdot\ \bar{\partial}_Y \text{ on } \Gamma(Y, \mathbb{C} \oplus \overline{T^*_{\mathbb{C}} Y}) \end{cases}$

<u>claim</u> 上のとき $\operatorname{ind} \hat{D} = \cancel{\operatorname{ind} D_X} - \dim \operatorname{Ker} D^*_{0,X}$

$$\because)\ \operatorname{Ker}_0 \hat{D}_0 = (\ker D_{0,X} \otimes \overbrace{\ker_0 \bar{\partial}_Y}^{=0}) \oplus (\operatorname{Ker} D^*_{0,X} \otimes \overbrace{\ker_0 \bar{\partial}_Y^*}^{=0}) = 0$$

$$\ker_b \hat{D}_0^* = (\ker D_{0X} \otimes \underbrace{\ker_b \bar{\partial}_Y^*}_{=0}) \oplus (\operatorname{Ker} D^*_{0X} \otimes \underbrace{\ker_b \bar{\partial}_Y}_{=\mathbb{C}}) = \operatorname{Ker} D^*_{0X} \quad //$$

(UeⅡからもでる)

<u>claim</u>

$$\underset{\underset{X}{\shortparallel}}{H_2(\mathbb{CP}^2, \mathbb{Z})} \ni \underset{\underset{c_1(\Lambda^2 T_{\mathbb{C}} X)}{\Big\updownarrow \text{P.D}}}{3} = [\Sigma] \Rightarrow \text{genus}(\Sigma) \geq 1$$

<u>claim のいいかえ</u>

$$\begin{array}{c} \Lambda^2 T_{\mathbb{C}} X \\ \downarrow \big\uparrow s \\ X \end{array}$$

0と横断的 $\Sigma = s^{-1}(0) \Rightarrow \text{genus}(\Sigma) \neq 0$

X上で Dolbeault op $\bar{\partial}_X$

$$\nu \overset{\text{normal}}{\cong} \Lambda^2 T_{\mathbb{C}} X|_{\Sigma}$$
$$\downarrow$$
$$\Sigma$$

仮に、s(あ, 2Σ) が flat とする.

$$T_{\mathbb{C}}\Sigma \oplus \nu \cong T_{\mathbb{C}} X|_{\Sigma}$$

Λ^2 をとると

$$T_{\mathbb{C}}\Sigma \otimes \nu \cong \Lambda T_{\mathbb{C}} X|_{\Sigma}$$

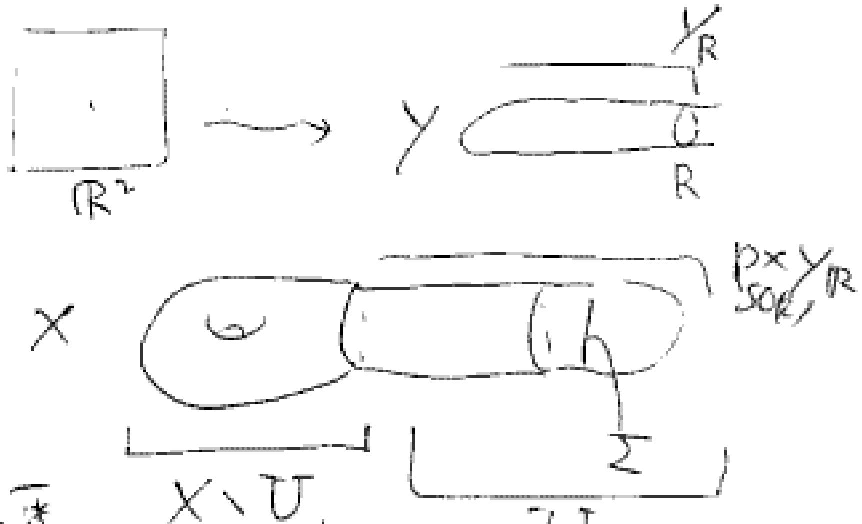
$$\nu = \Lambda^2 T\, T_{\mathbb{C}} X|_{\Sigma} \leftarrow \text{同じ}$$

$\Longrightarrow$ $\boxed{T_{\mathbb{C}}\Sigma \text{は自明化をもつ}}$ ($\Longleftrightarrow \Sigma =$ トーラス)

一般の場合

$$\begin{array}{ccc} L & = & P\times \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \Sigma & = & \Sigma \end{array}$$

$$\begin{array}{c} P \\ \downarrow{\scriptstyle SO(2)} \\ \Sigma \end{array}$$



和公式

$$\mathrm{ind}\begin{pmatrix} 0 & \bar\partial_X^* \\ \bar\partial_X & 0 \end{pmatrix} = \underset{\equiv 0\,(2)}{\mathrm{ind}\begin{pmatrix} 0 & \bar\partial_{X\setminus U}^* \\ \bar\partial_{X\setminus U} & 0 \end{pmatrix}} + \mathrm{ind}\begin{pmatrix} 0 & \bar\partial_U^* \\ \bar\partial_U & 0 \end{pmatrix} + \underset{\equiv 0\,(2)}{\dim \mathrm{Ker}\, A_{+\infty}}$$

積公式

$$\mathrm{ind}\begin{pmatrix} 0 & \bar\partial_U^* \\ \bar\partial_U & 0 \end{pmatrix} = -\dim \mathrm{Ker}\, \bar\partial_{\Sigma, K^{\frac{1}{2}}}$$

$K^{\frac{1}{2}} = (T_{\mathbb{C}}^*\Sigma)^{\frac{1}{2}}$

Fact
- $X\setminus U$ 上の op ← 4 dim
- ∂U 上の op $A_{+\infty}$ ← 3 dim

は 四元数体上のベクトル空間上の作用素

& 有限次元なら偶数

結論

$$\mathrm{ind}\begin{pmatrix} 0 & \bar\partial_X^* \\ \bar\partial_X & 0 \end{pmatrix} \underset{(\mathrm{mod}\,2)}{\equiv} -\dim \mathrm{Ker}\, \bar\partial_{\Sigma,(T_{\mathbb{C}}^*\Sigma)^{\frac{1}{2}}}$$

トーラス　$T_{\mathbb{C}}^*\Sigma$ の自明化が与えられた

$\rightsquigarrow (T_{\mathbb{C}}^*\Sigma)^{\frac{1}{2}}$ ← 自明!!

このとき 左辺 ≡ 1 (Fact 左辺 = 1)

仮に $\Sigma = S^2 = \mathbb{P}^1$ としよう

$\mathrm{Ker}\, \bar\partial_{\mathbb{P}^1, (T^*\mathbb{P}^1)^{\frac{1}{2}}} = 0$