2001.02.21

Y. Kametani I

I D-inv.
II structure thm.

§0. model

G cpt Lie gr $\curvearrowright Z$ free V: G-rep. sp.

$f: Z \to V$ G-equiv.
$\quad\ 0 \in V$ reg. value

$M \equiv f^{-1}(0)/G$ smooth mfd
$\Rightarrow \dim M = \dim Z - \dim V - \dim G$

$$0 \to \mathfrak{g} \xrightarrow{dL_g} T_xZ \xrightarrow{df_x} V \to 0 \quad \text{complex}$$

$$\dim M = -h^0 + h^1 - h^2 = \chi$$

§1. X: closed ori. 4-mfd. $\pi_1 = 1$ g: R-metric

$E \xrightarrow[(,)]{\mathbb{C}^2} X \quad P = F_r(E) \xrightarrow{SU(2)} X$

$\mathcal{A}_E = \{A : su(2)\text{-conn. on } P\}$

$\quad = \{\nabla : \Omega^0(E) \to \Omega^1(E) \text{ cov. der.}\}$

$\mathcal{G}_E = \mathrm{Aut}(E) = \left\{ u : \begin{array}{ccc} E & \xrightarrow{\cong} & E \\ \downarrow & \circlearrowleft & \downarrow \\ X & = & X \end{array} \text{ isom} \quad \det u = 1 \right\}$

$\mathcal{G}_E \curvearrowright \mathcal{A}_E \qquad \{\pm 1\} \curvearrowright \mathcal{A}_E$ trivial

$$u \cdot \nabla \equiv u \cdot \nabla \cdot u^{-1} \left(\overset{\text{local}}{=} -du \cdot u^{-1} + uAu^{-1} \right)$$

$\mathcal{G}_E \curvearrowright \Omega^*(\mathrm{ad}\, E \coloneqq \mathrm{Herm}_0(E))$ ← traceless

$u \cdot s \coloneqq u s u^{-1}$

$$F^+ : \mathcal{A}_E \to \Omega^+(\mathrm{ad}\, E) \qquad \Omega^2 = \underset{+1}{\Omega^+} \oplus \underset{-1}{\Omega^-} \quad (*_g^2 = 1)$$

$$\mathcal{A}_E = A + \Omega^1(\mathrm{ad}\, E) \ni A \mapsto F_A^+ = (dA + \tfrac{1}{2}[A \wedge A])^+$$

$M_E(g) \coloneqq (F^+)^{-1}(0)/\mathcal{G}_E$ ASD-moduli sp.

$$0 \to \underset{\mathrm{Lie}(\mathcal{G}_E)}{\Omega^0(\mathrm{ad}\, E)} \xrightarrow{d_A} \Omega^1(\mathrm{ad}\, E) \xrightarrow{d_A^+} \Omega^+(\mathrm{ad}\, E) \to 0$$

A-H-S complex

$$\dim M_E = -h^0 + h^1 - h^2$$
$$\underset{\text{index thm}}{=} 8c_2(E) - 3(1 - \underset{0}{b^1} + b^+)$$

- $M_E(g)$ は smooth for a generic g
- $\mathcal{G}_E/\{\pm 1\} \curvearrowright \mathcal{A}_E$ はほとんどの点で free

§2. D-inv.

$$\mathcal{A}_E \times E \xrightarrow{\mathbb{C}^2} \mathcal{A}_X \times X \qquad \mathcal{G}_E$$

$\{-1\} \not\subset \tilde{\mathcal{G}}_E \coloneqq \{u \in \mathcal{G}_E \mid u(x_0) = \mathrm{id}\}$ $(x_0 \in X)$ $\subset \mathcal{G}_E$.

$$\tilde{\mathbb{E}} \coloneqq \mathcal{A}_E \times_{\tilde{\mathcal{G}}_E} E \xrightarrow{\mathbb{C}^2} \tilde{\mathcal{B}}_E \times X \quad SU(2)\text{ b}$$

$\tilde{\mathcal{B}}_E = \mathcal{A}_E/\tilde{\mathcal{G}}_E$

$c_2(\tilde{\mathbb{E}}) \in H^2(\tilde{\mathcal{B}}_E \times X)$

$\Sigma \subset X$ embedded surface

$\not{\partial} : \Gamma(S^+) \to \Gamma(S^-)$: Dirac op.

$A \in \mathcal{A}_E \to A|_\Sigma \in \mathcal{A}_{E|\Sigma}$

$\not{\partial}_{A|\Sigma} : \Gamma(E|_\Sigma \otimes S^+) \to \Gamma(E|_\Sigma \otimes S^-)$ Dirac op.

$$\begin{array}{ccc} \mathbb{D} : \mathcal{A}_E \underset{\hat{\mathcal{G}}_E}{\times} \Gamma(E|_\Sigma \otimes S^+) & \longrightarrow & \mathcal{A}_E \underset{\hat{\mathcal{G}}_E}{\times} \Gamma(E|_\Sigma \otimes S^-) \\ \| & & \| \\ \Gamma(\hat{\mathbb{E}}|_\Sigma \otimes S^+) & & \Gamma(\hat{\mathbb{E}}|_\Sigma \otimes S^-) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{B}}_{E|\Sigma} \times X & & \tilde{\mathcal{B}}_{E|\Sigma} \times X \end{array}$$

$\mathrm{ind}\, \mathbb{D}_\Sigma \in K^0(\tilde{\mathcal{B}}_E)$

$$\mathrm{ch}(\mathrm{ind}\, \mathbb{D}_\Sigma) \underset{\text{index thm.}}{=} \mathrm{ch}(\hat{\mathbb{E}}|_{\underset{?}{\tilde{\mathcal{B}}_\Sigma} \times \Sigma})\, \hat{A}(\Sigma) / [\Sigma]$$

$$= -\, c_2(\hat{\mathbb{E}}) / [\Sigma]$$

$$\tilde{\mathcal{L}}_\Sigma \equiv \det(\mathrm{ind}\, \mathbb{D}_\Sigma) \xrightarrow{\mathbb{C}} \tilde{\mathcal{B}}_E$$

$$c_1(\tilde{\mathcal{L}}_\Sigma) = c_2(\hat{\mathbb{E}}) / [\Sigma] = \tilde{\mu}([\Sigma])$$

$\tilde{\mu} : H_2(X) \to H^2(\tilde{\mathcal{B}}_E)$

$(-1) \curvearrowright \tilde{\mathcal{L}}_\Sigma$ act as $(-1)^N$

N: $\mathrm{ind}\, \not{\partial}_{A|\Sigma} \underset{\text{index thm}}{=} 0$

$$\mathcal{L}_\Sigma \equiv \widetilde{\mathcal{L}}_\Sigma / \underset{\substack{\| \\ SU(2)/\{\pm 1\}}}{SO(3)} \xrightarrow{\mathbb{C}} \mathcal{B}_{\mathbb{E}} = \widetilde{\mathcal{B}}_{\mathbb{E}} / SO(3)$$

<u>Def</u> $\mu([\Sigma]) = c_1(\mathcal{L}_\Sigma) \in H^2(\mathcal{B}_{\mathbb{E}})$

- μ は $\alpha = [\Sigma]$ のみよる.
  Σ の spin str によらない.
  $\left(\because \ \tilde{\mu}([\Sigma]) = c_2(\widetilde{\mathbb{E}})/[\Sigma] \right)$
- μ は Σ の nbd で決まる.

一方.

$$\begin{array}{ccc} \mathrm{ad}\,\widetilde{\mathbb{E}} & \xrightarrow{\mathbb{R}^3} & \widetilde{\mathcal{B}}_{\mathbb{E}} \times X \\ \big\downarrow SU(2) & & \big\downarrow SU(2) \\ \mathbb{E}^{ad} & \xrightarrow{\mathbb{R}^3} & \mathcal{B}_{\mathbb{E}} \times X \end{array}$$

$$\mu : \begin{array}{ccc} H_0(X) & \to & H^4(\mathcal{B}_{\mathbb{E}}) \\ \Psi & & \Psi \\ [x] & \longmapsto & -\frac{1}{4} p_1(\mathbb{E}^{ad})/[x] \end{array}$$

- これも x の nbd のみによる.

$$D_X : \mathrm{Sym}\left(H_0 \oplus H_2(X)\right) \longrightarrow \mathbb{R}$$
$$x^a \quad \alpha \quad \longmapsto \langle \mu(x)^a \mu(\alpha)^b, [M_{\mathbb{E}}] \rangle$$
$$(2a + 4b : \dim M_{\mathbb{E}})$$

~~§~~ "$\langle\ ,\ \rangle$" を正当化する方法.

0. M_E : oriented $\Longleftarrow$ $H_2^+(X)$: orientation

1. ② $\mu(x), \mu(\alpha) \xleftrightarrow{\text{P.D.}} V_x, V_\alpha$: codim 2 submfd.

$$\mathcal{L}_I \xrightarrow{\mathbb{C}} \mathcal{B}_E\ (\supset \mathcal{M}_E)$$

$$\text{"}\langle\ ,\ \rangle\text{"} = \bigcap_{a\ni} V_x \cap \bigcap_{b\ni} V_\alpha \cap M_E \quad \text{cpt}$$

① $\mathcal{M}_E \subset \overline{\mathcal{M}}_E$ Uhlenbeck's cpt化

$$\begin{array}{ccccc} \mu : H_{2,0}(X) & \longrightarrow & H^2(\mathcal{B}_E) & \longrightarrow & H^{2,4}(\mathcal{M}_E) \\ \| & & \circlearrowleft & & \\ \exists\ \bar{\mu} : H_2(X) & & \longrightarrow & & H^{2,4}(\bar{\mathcal{M}}_E) \end{array}$$

$$\text{"}\langle\ ,\ \rangle\text{"} = \langle \bar{\mu}(x)^a \bar{\mu}(\alpha)^b, [\bar{\mathcal{M}}_E] \rangle$$

2. $b^+ = 0$ のとき

$$\mathcal{B}_E^{red} = \{[A] \mid \exists u A = A,\ u \neq \{\pm 1\}\}$$

$$\mathcal{M}_E^{red}(g) \equiv \mathcal{M}_E(g) \cap \mathcal{B}_E^{red}$$

$$= \{c \in H^2(X;\mathbb{Z}) \mid c^2 = -c_2(E)\} / \{\pm 1\}$$

$\hat{\mu}(a) \in H^2(\mathcal{B}_E)$ defined

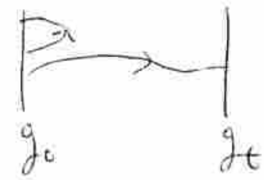
$SO(3)$

$\mu(a) \overset{?}{=}$

$b^+ \geq 1 \Rightarrow \mathcal{M}_E^{red}(g) = \phi$ for a generic g

$b^+ \geq 2 \Rightarrow \bigcup_{0 \leq t \leq 1} \mathcal{M}_E^{red}(g_t) = \phi$ for generic $\{g_t\}_{0 \leq t \leq 1}$

このとき, D_X が g によらない!



3. $c_2(E)=0 \Leftrightarrow E=\mathbb{C}^2$

$[\theta : \text{trivial}] \in \mathcal{M}_E$

これを避けるため $c_2(E) \geq \frac{1}{4}(3b^+ + 5)$ とする.

4. $E \xrightarrow[(\,,\,)]{\mathbb{R}^3} X$, $SO(3)$-b.

$\rightsquigarrow \dim \mathcal{M}_E = -2p_1(E) - 3(1-b^1+b^+)$

$\rightsquigarrow D_{X,c} : \mathbb{A}(X) \to \mathbb{Q} \quad \left(\begin{array}{l} c \in H^2(X;\mathbb{Z}) \\ c \equiv w_2(E) \end{array} \right)$

5. $\check{X}$: closed 4-orbifold $\quad$ $\check{X}$ x_0 $\quad$ $D^4/(\mathbb{Z}/p)$

$\check{E}[i] \to \check{X}$ orbi-$\underset{SO(3)}{SU(2)}$-b

$\cup$

$D^4 \times \mathbb{C}^2 / \mathbb{Z}/p \to D^4/(\mathbb{Z}/p)$

$\zeta^i \to \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{pmatrix} \quad \left(0 \leq i \leq \left[\frac{p}{2}\right] \right)$

$\tilde{\mathcal{M}}_{\check{E}[i]} = F^{+\,-1}(0) / \mathcal{G}_E$

$\partial : \tilde{\mathcal{M}}_{\check{E}[i]} \to S^2[i]$ 境界値写像

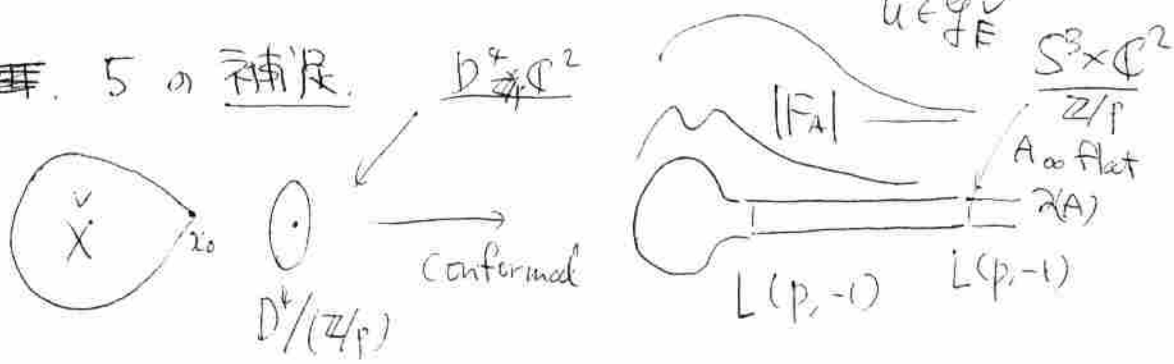
$= \begin{cases} 1\text{点} & (i=0) \\ S^2 \cong SU(2)/S^1 & (i \neq 0) \end{cases} \quad (p : \text{odd})$

$\rightsquigarrow D_{\check{X}}[i] : \mathrm{Sym}(H_0 \oplus H_2(\check{X})) \to \mathbb{Q}$

2001.02.22

Y. Kametani II

~~第~~. 5 の補足.



$\int_{\check X} |F_A|^2 < \infty$ $\qquad\qquad$ $\int |F_A|^2 < \infty$

$\partial : \tilde{\mathcal{M}}_{\check E}[i] \longrightarrow \frac{SU(2)/\{\pm 1\}}{S^1} \qquad i : \mathbb{Z}/4p \longrightarrow U(2)$ hom

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \longrightarrow \begin{pmatrix} \zeta^{i} & 0 \\ 0 & \zeta^{-i} \end{pmatrix}$

$$\| $$

$$\frac{SO(3)}{S^1} = S^2$$

$M_{\check E}[i] = F_+^{-1}(0)/\mathcal{G}_{\check E}$

$\qquad = \hat{M}_{\check E}[i]/SO(3) \qquad S^2[i] \quad (i = 0, \frac{p}{2})$

$\qquad\qquad \| \qquad\qquad\qquad \{[0]\} \qquad (i=0)$

$\qquad \frac{SU(2)}{\{\pm 1\}} \qquad\qquad \{[\frac{p}{2}]\} \qquad (i = \frac{p}{2})$ (p: even)

$b^+(\check X) = 0$

$\Rightarrow \hat{\mu}(\alpha) \in H_{SO(3)}(\tilde{\beta}_{\check E[i]}) \longrightarrow H_{SO(3)}(\check{\mathcal{M}}_{E[i]})$

$\mu(\alpha) \in H(M_{E[i]}) \quad \overset{?}{=}$

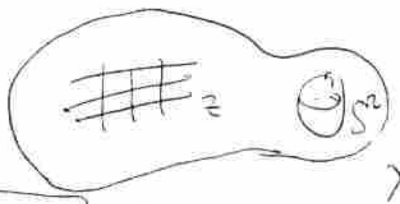
§3 embedded 2-sphere

$$\sigma = [S^2 \hookrightarrow X] \in H_2(X;\mathbb{Z}) \qquad \sigma^2 = -p \leq -3$$

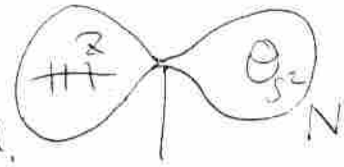
$$-p = -2k-1 \text{ odd.}$$

$$z \in \mathbb{A}(\sigma^\perp) = \mathbb{A}(H_1(X) \oplus \langle\sigma\rangle^\perp) \quad \mathrm{Sym}^*$$

$$D_X(\sigma^{2l-1} z) = ? \quad (1 \leq l \leq k)$$



$X \xrightarrow[\text{metricをかえる}]{b^+(X) \geq 2} N$



$\boxed{U_{z\sigma} \cap M_E(g)}$

$\partial N = L(p,-1)$

$\xrightarrow{\text{degenerate}}$

$\check{X}$ $\check{N}$

$$\int |F_{A_X}|^2 + \int |F_{A_N}|^2 = 8\pi^2 c_2(E)$$

fiber product

$$\bigcup_{0 \leq b \leq c_2(E)} \hat{M}_{\check{X}} \check{E}[i] \times_\partial \hat{M}_{\check{N}} \check{E}[i]$$

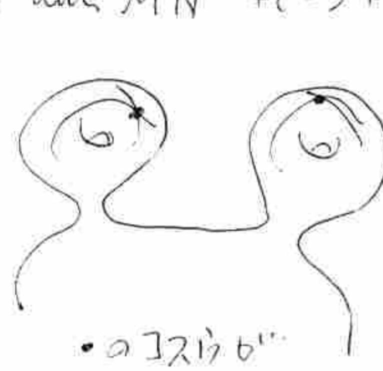
$SO(3)$

$c_2E - b$ $\quad$ b

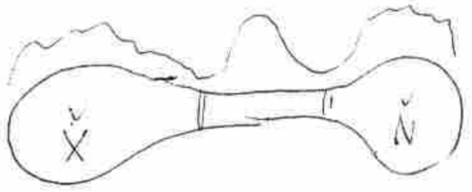
s.t. $\dim \tilde{M}_{\check{N}} = 4i - 3 + b \in \mathbb{Z}$

・のコスウは
かぞえにくい

$\xrightarrow{\text{metric 変形}}$



・のコスウが
かぞえやすい



2つコブ → 有限でとまる $8\pi^2$

3つコブ　　3/17

$$D_X(\sigma^{2l-1}z) = \sum_{i,b} \left\langle \mu(\sigma)^{2l-1}\mu(z), \left[\frac{\tilde M_{\check X} \# \tilde M_{\check N}}{SO(3)}\right]\right\rangle \quad ①$$

$$= \sum \left\langle \quad '' \quad , \left[\frac{\hat M_{\check X} \# \hat{\tilde M}_L \# \tilde M_{\check N}}{SO(3)}\right]\right\rangle$$

$$+ \sum + \cdots \text{ finite sum } \left(\int = 8\pi^2 c_2\right)$$

$$\begin{tikzcd} & \hat M_{\check X} \# \tilde M_{\check N} \arrow[dl, "\pi"'] \arrow[dr, "\pi_N"] & \\ \tilde M_{\check X} & & \hat M_{\check N} \end{tikzcd}$$

SO(3)-equiv.　$① = \langle \pi_*(\pi_N^*(\hat\mu(\sigma))^{2l-1}), [M_{\check X}]\rangle$

π_*: fiber 積分

model　G : cpt Lie grp $\curvearrowright \tilde X$ (free) $\curvearrowright \tilde Y$

$G \supset H$

$\tilde X \xrightarrow{f_X} G/H \xleftarrow{f_Y} \tilde Y$　G-equiv.

$\tilde a \in H_G^*(\tilde Y)$

$\mapsto$ $G \curvearrowright \tilde Z \equiv \tilde X \times_{G/H} \tilde Y$ ← fiber product　free

$\cancel{G}$

$$\begin{tikzcd} \tilde Z/G = Z \simeq \dfrac{EG\times\tilde Z}{G} \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & \dfrac{EG\times\tilde Y}{G} \arrow[d, "f_X"] \\ \tilde X/G = X \simeq \dfrac{EG\times\tilde X}{G} \arrow[r, "f_X"'] & \dfrac{EG\times G/H}{G} \simeq BH \end{tikzcd}$$

$$\pi_*(\pi_Y^*(\tilde a)) = f_X^*(\underbrace{f_{Y*}(\tilde a)}_{\in H^*(BH)})$$

$g \in G$

$$\mapsto \begin{tikzcd} \tilde X \supset f_X^{-1}(g) \arrow[d, "G"'] \arrow[r, phantom, "="] & EH\times f_X^{-1}(g) \arrow[r, "f_X"] \arrow[d, "H"] & EH\times\{g\} \arrow[d, "H"] \\ X \supset f_X^{-1}(g) \arrow[r, phantom, "\cong"] & \frac{EH\times f_X^{-1}(g)}{H} \arrow[r, "f_X"] & BH\times\{g\} = BH \end{tikzcd}$$

$$f_X^*(f_{Y*}(\tilde a)) = \sum_i a_i d_i(f_X^{-1}(g) \xrightarrow{H} f_X^{-1}(g)/H)$$

$$(\in H^*(BH) = \mathbb{Q}[d_1, \dots, d_n])$$

char. class

Ex), $G = SO(3)$, $H = S^1$

$\mapsto$ $f_{Y*}(\tilde a) = a c_1^{2l}$ $(H^*(BS^1) = \mathbb{Q}[c_1])$

$\mapsto$ $f_X^*(f_{Y*}(\tilde a)) = a\,(P(\tilde X \xrightarrow{SO(3)} X))^l$

① $= {}^\exists m \langle \mu(z)\mu(x)^{l-n-2b}, [M_{\tilde X}[i]] \rangle$

$(\because Fr(\mathbb{E}^{ad}) = \hat{B}_E \times X \xrightarrow{SO(3)} B_E \times X)$

$= m D_{\tilde X}[i](zx^{l-2i-2b})$

$$D_X(\sigma^{2l-1}z) = \sum_{i=1}^{l}\sum_{b=0}^{[\frac{l-i}{2}]} \underset{\underset{l}{\uparrow}}{r_{i,b}}\, D_X^{\vee}[i](z x^{l-i-2b}) \quad (1 \le l \le k)$$

$$\iff \begin{cases} D_X(\sigma z) = r_{1,1,0}\, D_X^{\vee}[1](z) \\ D_X(\sigma^3 z) = r_{2,1,0}\, D_X^{\vee}[1](zx) + r_{2,2,0}\, D_X^{\vee}[2](z) \\ \quad \vdots \\ D_X(\sigma^{2k-1}z) = \cdots \end{cases}$$

$r_{1,1,0} \neq 0$ (X = elliptic surface を代入)

$r_{2,2,0} \neq 0$ (〃)

<u>Claim</u> $D_X^{\vee}[k](z) = D_{X,\sigma}^{\vee}(z)$ (gluing)

<u>Th</u> $D_X(\sigma^{2k-1}z) = B_{0,0,k} D_{X,\sigma}(z) + \sum_{i=1}^{k-1}\sum_{b=0}^{[\frac{k-i}{2}]} B_{i,b,k}\, D_X(\sigma^{2i-1} x^{k-i-2b} z)$

$p = -2k$ も同様

$\rightsquigarrow$

$$X : \text{simple type} \overset{\text{def}}{\iff} D_X(x^2 z) = 4 D_X(z) \quad (\forall z \in \mathbb{A}(X))$$

$$\iff D_X\left(x\left(1+\frac{x}{2}\right)z\right) = \hat{D}_X(xz)$$

$$= 2 D_X\left(\left(1+\frac{x}{2}\right)z\right) = 2\hat{D}_X(z)$$

$$\mathbb{D}_X : H_2(X) \to \mathbb{R} \quad \text{formal}$$

$$\alpha \mapsto \hat{D}_X(e^{\alpha}) = \sum_{n=0}^{\infty} \frac{\hat{D}_X(\alpha^n)}{n!}$$

determines $D_X : \mathbb{A}(X) \to \mathbb{Q}$

Q : int. form of X

$$e^{\frac{Q}{2}} : H_2(X) \to \mathbb{R}$$

$$\mathbb{K}_X \equiv e^{-\frac{Q}{2}} \mathbb{D}_X$$

$$\xrightarrow[\text{Th より}]{} (\partial_u^2 - 1^2)(\partial_u^2 - 3^2) \cdots (\partial_u^2 - (2k-1)^2)\mathbb{K}_X \equiv 0$$

$$\rotatebox{90}{$\geqq$}$$

$\mathring{\Delta}_{u,k+p}$ (p: pos. int のコスち?)

$$\left(u = [S^2 \to X],\ u^2 = -2k-1 \leqq -3\right)$$

$$\pi_1 = 1 \quad H_2(X;\mathbb{Z}) = \pi_2(X) = \mathbb{Z}\langle u_i = [S^2 \to X] \rangle_{1 \leq i \leq b_2}$$

$$\mathbb{K}_X = \sum_{j_1 \cdots j_b} \Phi_{j_1 \cdots j_b}\, e^{r_{1,j_1} u_1^* + \cdots + r_{b,j_b} u_b^*}$$

$$(r_{1,j_1} \cdots \in \mathbb{Z})$$

Str. Thm X : simple type $\pi_1 = 1$

$$\Rightarrow \mathbb{D}_X = e^{\frac{Q}{2}} \left(\sum_{s=1}^{p} a_s e^{K_s} \right)$$

$$(a_s \in \mathbb{Q}, K_s \in H^2(X; \mathbb{Z}))$$

例 $X = K3$ surface

$$\mathbb{D}_X = e^{\frac{Q}{2}}$$

$$\mathbb{D}_{E(n)} = e^{\frac{Q}{2}} \sinh^{n-2}(f) \qquad 1 \leq i \leq n$$

$$\left(\underset{\text{no multi fiber}}{E(n) \xrightarrow{f} \mathbb{P}^1} \quad \chi(E(n)) = 12n \right)$$

Witten conj

$$\mathbb{D}_X = e^{\frac{Q}{2}} \left(\sum_{\substack{K \in H^2(X;\mathbb{Z}) \\ K \equiv w_2(TX)}} SW(K) e^K \right)$$

(ref.) K-M JDG 41

F-S JDG 42