M. Ue II

$g \curvearrowright X^n \quad n = 4k \quad X^g = \{ x \in X \mid gx = x \}$

orientable とする

$\underset{\substack{N^g \\ \text{normal 束}}}{X^g \hookrightarrow X} \qquad TX|_{X^g} \cong TX^g \oplus N^g$ $\quad N_\theta \cong \oplus L^j_\theta$

$N^g \cong \oplus N^g_\theta \quad$ fiber上 $g = e^{i\theta}$ ($\theta = \pi$ だとちょっとマズイ)

$$\mathrm{Sign}(g, X) = \mathrm{tr}\, g|H^+ - \mathrm{tr}\, g|H^-$$

$$= \frac{ch^g(\Lambda^+ - \Lambda^-)(N^g \otimes \mathbb{C})}{ch^g \Lambda_{-1}(N^g \otimes \mathbb{C})} \frac{ch^g(\Lambda^+ - \Lambda^-)(TX^g \otimes \mathbb{C})\, td(TX^g \otimes \mathbb{C})}{e(TX^g)} [X^g]$$

$$\frac{ch^g(\Lambda^+ - \Lambda^-)(N^g \otimes \mathbb{C})}{ch^g \Lambda_{-1} N^g \otimes \mathbb{C})} = \frac{ch^g(\Lambda^+ - \Lambda^-)(L^j_\theta \otimes \mathbb{C})}{ch^g \Lambda_{-}(L_\theta \otimes \mathbb{C})} = \frac{e^{-i\theta} e^{-x_j} - e^{i\theta} e^{x_j}}{(1 - e^{i\theta} e^{x_j})(1 - e^{-i\theta} e^{-x_j})}$$

$n = 4$

- $X^g = \{p\} \quad N^g_p \cong \mathbb{C} \oplus \mathbb{C}$

  $g(z_1, z_2) = (\underbrace{e^{i\theta_1} z_1}_{L_{\theta_1}}, \underbrace{e^{i\theta_2} z_2}_{L_{\theta_2}})$

$$\rightsquigarrow \prod_{k=1}^{2} \frac{e^{-i\theta_k} - e^{+i\theta_k}}{(1 - e^{i\theta_k})(1 - e^{-i\theta_k})} = \prod_{k=1}^{2} (-i) \cot \frac{\theta_k}{2} = -\cot \frac{\theta_1}{2} \cot \frac{\theta_2}{2}$$

- $X^g = F$ surface $p \in F$: $N_p \cong \underset{z}{\mathbb{C}} \quad gz = e^{i\theta} z \qquad TF \underset{N}{\hookrightarrow} TX$

$$\mathrm{Sign}(g, X) = \coth\left(x + i\frac{\theta}{2}\right)[F] \qquad c_1(N) = x$$

$$= \mathrm{cosec}^2 \frac{\theta}{2}\, x[F]$$

$$= \mathrm{cosec}^2 \frac{\theta}{2} [F]^2$$

<u>n=4</u>

$$\mathrm{Sign}(g, X) = \begin{cases} \mathrm{Sign}\, X & g = 1 \\ -\sum_p \cot\frac{\theta_1(p)}{2}\cot\frac{\theta_2(p)}{2} + \sum_F \mathrm{cosec}^2\frac{\theta(F)}{2}[F]^2 \end{cases}$$

$g|N_F \quad g(z_1) = (e^{\theta(F)} z_1)$

$N_p \cong T_p X \quad g|N_p \quad g(z_1, z_2) = (e^{\theta_1(p)} z_1, e^{\theta_2(p)} z_2)$

$\pi : X \to X/G \qquad G$: finite. <u>Semifree</u>

$$\mathrm{Sign}(X/G) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Sign}(g, X)$$

($G \curvearrowright V$ vec.sp. $\quad \dim V^G = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}\, g$

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ & \searrow^{\pi} & \cup\, i \\ & & V_G \end{array}$$

$v \longmapsto \frac{1}{|G|}\sum gv \qquad \left.\begin{array}{l} \pi i = \mathrm{id} \\ i\pi = f \end{array}\right\}$ ココカのtraceは等しい)

($g \curvearrowright H^{2k}_{\pm} \qquad H^*(X/G, \mathbb{Q}) \cong H^*(X, \mathbb{Q})^G$

$\cdot\ \mathrm{tr}\, g | H^{2k}_+ - \mathrm{tr}\, g | H^{2k}_-$)

"signature defect"

<u>dim X = 4</u> $\quad |G|\cdot\mathrm{Sign}(X/G) = \mathrm{Sign}X + \sum_p \mathrm{def}_p + \sum_F \mathrm{def}\, F$

$$\mathrm{def}_p = -\sum_{\substack{g\in G\\ g\neq 1}} \cot\frac{\theta_1(p)^g}{2}\cot\frac{\theta_2(p)^g}{2} \qquad \mathrm{def}_F = \sum_{\substack{g\in G\\ g\neq 1}} \mathrm{cosec}^2\frac{\theta_F^g}{2}[F]^2$$

$\langle g\rangle \cong \mathbb{Z}_n$ $\quad \mathrm{def}_F = \sum_{\substack{e^{in\theta}=1 \\ e^{i\theta}\neq 1}} \mathrm{cosec}^2\frac{\theta}{2}[F]^2 = \frac{n^2-1}{3}[F]^2$

p : isolated fixed point.

$g(z_1, z_2) = (e^{2\pi i/n} z_1, e^{2\pi i\ell/n} z_2)$ $\gcd(n, \ell) = 1$ とする.

$T_p X \cong \mathbb{C}^2$ $\quad \mathrm{def}_p = -\sum_{k=1}^{n-1} \cot\frac{\pi k}{n} \cot\frac{\pi k\ell}{n}$ Dedekind sum

$\circlearrowleft g^k$

(Dedekind sum
$s(\ell, n) = \frac{1}{4n}\sum_{k=1}^{n-1} \cot\frac{\pi k}{n}\cot\frac{\pi k\ell}{n}$)

$s(q, p) = \frac{1}{4p}\sum_{k=1}^{p-1} \cot\frac{\pi k}{p} \cot\frac{\pi kq}{p}$ $\quad \gcd(p, q) = 1$

$= \sum_{k=1}^{p} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{kq}{p}\right)\right)$ $\quad ((x)) = \begin{cases} 0 & x \in \mathbb{Z} \\ x - [x] - \frac{1}{2} & x \notin \mathbb{Z} \end{cases}$

Dedekind reciprocity

$$s(q, p) + s(p, q) = \frac{p^2 + q^2 + 1 - 3pq}{12pq}$$

$$s(1, p) = \frac{(p-1)(p-2)}{2}$$

- G-signature Th の応用. (Rochlin, Hsiang-Szczarba)

X 4-mfd $\quad \underset{\text{surface}}{\Sigma} \subset X \quad g(\Sigma)$ の評価.

$\xi = [\Sigma] \in H_2(X, \mathbb{Z}) \quad \underset{\substack{\vee \\ 1}}{m} \mid \xi$ とする. $\quad \underline{H_1(X, \mathbb{Z}) = 0}$ とする

$\pi : Y \to \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ Y/\mathbb{Z}_m}}{X} \quad \mathbb{Z}_m$ cover branched over Σ

$Y^{\mathbb{Z}_m} = \Sigma = \tilde{\Sigma}$

$n \doteq \max\{ n \in \mathbb{Z} \mid n > 0 \ \ n \mid m \}$

$U = \overline{X \setminus N} \qquad \pi_1 U \to H_1 U \cong \mathbb{Z}_n \longrightarrow \mathbb{Z}_m$

$\tilde{U} \xrightarrow[\mathbb{Z}_m]{} U$ unbranched cover.

$\exists\, \hat{N} \longrightarrow N \quad$ fiber $\cong D^2 \subset \mathbb{C}$

$z \longmapsto z^m$

$Y = \tilde{U} \cup \hat{N}$

g : generator of $\mathbb{Z}_m \qquad \underset{g^m = 1}{g} \circlearrowright \underset{\langle\, ,\, \rangle}{H^2(Y, \mathbb{C})} \cong \bigoplus_{r=0}^{m-1} L_r \circlearrowleft g$

L_r 上 $g = e^{\frac{2\pi i r}{m}}$

$\sigma_r = \operatorname{sign} L_r = \dim_{\mathbb{C}} L_r \cap H^+ - \dim_{\mathbb{C}} L_r \cap H^-$

$$\operatorname{Sign}(g^s, Y) = \sum_r \operatorname{tr} g^s|_{L_r \cap H^+} - \operatorname{tr} g^s|_{L_r \cap H^-}$$

$$= \sum_{r=0}^{m-1} \sigma_r \zeta^{sr} \qquad \zeta = e^{\frac{2\pi i}{m}}$$

逆に $\sigma_r = \frac{1}{m}\sum_{s=0}^{m-1} \mathrm{sign}(g^s, Y)\,\zeta^{-sr}$
$= \mathrm{Sign}\,X - \frac{2r(m-r)}{m^2}[\Sigma]^2$

$\left(\begin{array}{l}\mathrm{Sign}\,X = \frac{1}{m}\sum_g \mathrm{sign}(g, Y) \\ \mathrm{Sign}(1, Y) = \mathrm{Sign}\,Y\end{array}\right)$
$\left([\tilde{\Sigma}]^2 = \frac{[\Sigma]^2}{m}\right)$

m: prime power $H_1(X,\mathbb{Z}) = 0$

$\rightsquigarrow b_1 Y = 0$ $\quad \chi(Y) = m\chi(X) - (m-1)\chi(\Sigma)$

$\chi(Y) = 2 + \underbrace{b_2(Y)}_{= \sum \mathrm{rk} L_r}$, $\quad \chi(\Sigma) = 2-2g$

$b_2(Y) \geq |\mathrm{sign}\,Y|$

$\left(\mathrm{rk} L_0 = \dim H^2(Y,\mathbb{C})^g = \dim H^2(X,\mathbb{C}) = b_2(X)\right)$

(Rochlin) $\mathrm{rk} L_r = b_2(X) + 2g(\Sigma) \geq |\sigma_r| \Rightarrow g$ の評価.

$\Rightarrow \quad g \geq \left| \frac{r(m-r)}{m^2}\xi^2 - \frac{\mathrm{Sign}\,X}{2} \right| - \frac{b_2(X)}{2}$

$m = p^k$ (odd prime), $r = \left[\frac{m}{2}\right]$

$p^k | \xi \Rightarrow g(\Sigma) \geq \left| \frac{(n^2-1)\xi^2}{4n^2} - \frac{\mathrm{Sign}\,X}{2} \right| - \frac{b_2(X)}{2}$

Kotschick-Matic '95 $\quad 2^r | [\Sigma] \quad PD\frac{[\Sigma]}{2^r} \equiv w_2(X) \mod 2$
$H_1(X,\mathbb{Z}) = 0$

$\left(\begin{array}{l}\cdot \text{ Donaldson} \\ \quad \mathrm{sign}\,Y \neq 0 \Rightarrow b_2^{\pm} \geq 3 \\ \cdot \text{ Furuta } \frac{10}{8}\ \pi_1 \\ \quad \Rightarrow \text{better estimate}\end{array}\right)$

$Y \to X$
spin
$\mathrm{Sign}\,Y = 4792$
$b_2 Y = 6292$.
$b_1 Y = 0$