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Dirac operator

X^n spin (spin^c) str を考える

$$\begin{array}{cc} Spin(n) & P_{spin(n)} \\ \downarrow & \downarrow \\ SO(n) & P_{SO(n)} \end{array}$$

Local chart U_α $\mathbb{Z}_2 = \{\pm 1\}$

$$\begin{tikzcd} U_\alpha \cap U_\beta \arrow[r, "\hat{g}_{\alpha\beta}"] \arrow[dr, "g_{\alpha\beta}"'] & Spin(n) \arrow[d] \\ & SO(n) \end{tikzcd}$$

$w_2 \in H^2(X, \mathbb{Z}_2)$

$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = \{\pm 1\} \subset \mathbb{Z}_2$

$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$

$w_2 = 0 \Leftrightarrow$ spin

$Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} S^1$ $\{(1,1), (-1,-1)\}$

$U_\alpha \cap U_\beta \longrightarrow Spin^c(n)$

$\hat{g}_{\alpha\beta} \in Spin$, $z_{\alpha\beta} \in S^1$

$(\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha}, z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}) = \pm(1,1)$ のとき Spin^c str が入る. σ

$U_\alpha \cap U_\beta \xrightarrow[z_{\alpha\beta}^2]{} S^1 \Rightarrow \underline{\det \sigma}$ $\mathbb{C}$-bundle $c_1(\det\sigma) \equiv w_2 \bmod 2$

$\exists$ Spin^c str $\Leftrightarrow \exists$ w_2 の integral lift.

Dirac operator $(n = 2l)$

$\not{D} : C^\infty(S) \to C^\infty(S)$

X spin $Spin(n) \xrightarrow{\rho} End(\Delta)$ $S^\pm = P_{spin(n)} \times_\rho \Delta^\pm$

$\Delta = \Delta^+ + \Delta^-$ $S = S^+ + S^-$

$T^*X \xrightarrow{c} End(S)$

X metric ∇ Levi-Civita conn ($so(n)$-valued 1-form)

$S^\pm$ 上の conn が induce される.
$\tilde{\nabla}$

$$\begin{array}{ccc} \mathrm{spin}(n) & \xrightarrow{\rho_\pm} & \mathrm{End}(\Delta^\pm) \\ \downarrow & & \\ \mathfrak{so}(n) & & \end{array}$$

$$\not{D} : C^\infty(S) \xrightarrow{\tilde{\nabla}} C^\infty(T^*X \otimes S) \xrightarrow{c} C^\infty(S)$$

$$\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : \begin{array}{c} C^\infty(S^+) \\ \oplus \\ C^\infty(S^-) \end{array} \longrightarrow C^\infty(S) \qquad D^* = D \qquad (D^*)$$

$$D^+ : C^\infty(S^+) \longrightarrow C^\infty(S^-)$$

$$\mathrm{ind}\, D^+ = \dim_{\mathbb{C}} \mathrm{Ker}\, D^+ - \dim_{\mathbb{C}} \mathrm{Ker}\, D^-$$

$$= (-1)^\ell \frac{\mathrm{ch}(S^+ - S^-)\, \mathrm{td}(TX \otimes \mathbb{C})}{e(TX)} [X]$$

$V^n \quad n = 2\ell \qquad V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$

$(T_x X) \qquad (e_1\ f_1\ e_2\ f_2\ \cdots\ e_\ell\ f_\ell)$ local に $x_1, y_1 \cdots x_\ell\, y_\ell$ と思う

$e_i \xrightarrow{J} f_i \xrightarrow{J} -e_i$

$\Delta = \Lambda^{*,0} V \qquad \Delta^+ = \Lambda^{\mathrm{even},0} V, \quad \Delta^- = \Lambda^{\mathrm{odd},0} V$

$\Lambda^{*,0} V \cong \Lambda^{0,*} V^*$

$$dz_k = dx_k + i\, dy_k \qquad \left\{ \begin{array}{l} c(dz_k) = \sqrt{2}\, e(dz_k) \quad \text{外積} \\ c(d\bar{z}_k) = -\sqrt{2}\, \iota(d\bar{z}_k) \quad \text{内部積} \end{array} \right.$$

$$c : T^*M \to \mathrm{End}(S)$$

$$\mathrm{Cl}\, V \otimes \mathbb{C} \curvearrowright \mathrm{End}(S)$$

canonical spin^c str

$U(l) \dashrightarrow^{\tilde{\xi}} Spin(n) \times_{\mathbb{Z}_2} S^1 = Spin^c(n)$

$\tau \mapsto (\tilde{\tau}, \det \tau)$

$SO(n)$

$S^1 \xrightarrow{\ z\ } S^1$, $z \mapsto z^2$

X cx str をもつとき $K = \Lambda^l T_c^* \to K^{\frac{1}{2}}$ が存在 $\Leftrightarrow$ spin

$S^+ - S^- = (\Lambda^{0,even} - \Lambda^{0,odd}) \otimes K^{\frac{1}{2}}$

"splitting principle"

X complex $TX = P_1 \oplus \cdots \oplus P_l$

$L_1 \cdots L_l$

$c_1(L_j) = x_j$

$$ch(S^+ - S^-) = \prod_j (1 - e^{x_j}) e^{-\frac{x_j}{2}} = \prod (e^{-\frac{x_j}{2}} - e^{\frac{x_j}{2}})$$

$$\operatorname{ind} \not{D}^+ = \hat{A}[X] \qquad \hat{A} = \prod \frac{\frac{x_j}{2}}{\sinh \frac{x_j}{2}} = 1 - \frac{p_1}{24} + \cdots$$

$\dim X = 4$

$$\hat{A}[X] = -\frac{p_1}{24}[X] = -\frac{\operatorname{sign} X}{8}$$

- Vanishing Th

Lichnerowicz X spin

$$D^2 = \nabla^* \nabla + s \quad \text{scalar curvature}$$

Th $s > 0$ a metric をもつ

$\Longrightarrow \hat{A}(X) = 0$

$D\varphi = 0 \Rightarrow$

$0 = (D^2\varphi, \varphi)$

$= (\nabla^*\nabla\varphi, \varphi)$

$+ (s\varphi, \varphi)$

$\vee\!\!|$

0

(Atiyah-Hirzebruch)

X non-trivial S^1-action をもつ $\Rightarrow \hat{A}(X)=0$

S^1-action が spin str を保つとき $g \in S^1$ $\mathrm{ind}_{S^1} \not{D}^+ = 0$

$\mathbb{C}[\![\infty]\!]$ $P_{spin} \supset Spin(g,X) = tr_g\, \mathrm{ind}_{S^1} = 0$ $\qquad X^g \overset{N}{\hookrightarrow} X$

$g \in S^1$

$\overline{\langle g \rangle} = S^1$

$$Spin(g,X) = (-1)^e \frac{ch_1^g(S^+-S^-)(N^g\otimes\mathbb{C})\, ch_1^g(S^+-S^-)(TX^g\otimes\mathbb{C})\, td(TX^g\otimes\mathbb{C})}{ch^g(\Lambda_{-1}N^g\otimes\mathbb{C})\, e(TX^g)} [X^g]$$

V-mfd の index th.
(orbifold) (Kawasaki)

X n-dim V-mfd
$x \in X$

x の nbd $U \cong \tilde{U}_{\tilde{x}}/G_{\tilde{x}} \cong \hat{D}^n/G$ $\qquad G \subset SO_n$ finite subgrp

$x \longleftrightarrow \tilde{x} \longleftrightarrow 0$

$|G| \neq 1 \Leftrightarrow x$ は singular pt

V
- metric form
- bundle connection
- $\not{D}$
- sign
- index
- ⋮

defined.

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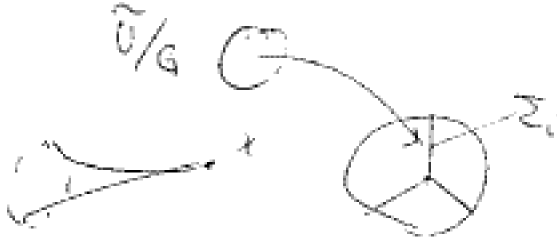
(例1) 孤立特異点の場合.



$U \cong \left(S^{n-1}/G\right)$ の cone

$G \subset SO_n$ free

$$\mathrm{ind}\, D = (-1)^{\dim} \mathrm{ch}\, \sigma(D)\, \mathrm{td}(TX \otimes \mathbb{C})[TX]$$

$$+ \sum \frac{(-1)^{\dim \Sigma_i}}{m_i} \left(\mathrm{ch}^{\Sigma_i} g^{\Sigma_i}(x)\ \mathrm{Tr}\, \Sigma_i \right)$$ ← $\Sigma_i X$ からの寄与.

multiplicity

Kawasaki のキゴウ

p からの寄与 $\quad U \cong \tilde{U}/G$

$\qquad\qquad\qquad p \quad \tilde{p}$

$$\sum_{(h) \in C(G)} \frac{1}{m_G^h} \frac{\mathrm{ch}^h j^* \sigma(D)}{\mathrm{ch}^h \Lambda_{-1}(N' \otimes \mathbb{C})}$$

$Z_G(h) := \{ g \in G \mid gh = hg \}$

$m_G^h = \# \{ g \in Z_G(h) \mid g\tilde{p} = \tilde{p} \}$

G の conjugacy class

$h \neq 1$

$$\mathrm{sign}(X/G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{sign}(g, \tilde{X})$$

特別な場合. $\dim X = 4$

孤立特異点 x_j : $j = 1, \cdots, k$

$U_{x_j} \cong D^4/\mathbb{Z}_{p_j}$ — generator g

$g(z_1, z_2) = (\zeta z_1, \zeta^{q_j} z_2) \quad \zeta = e^{\frac{2\pi i}{p_j}}$

$U_{x_j} \cong L(p_j, q_j)$ の cone